Title of lecture course: Introduction to comparison theorems in Riemannian geometry

Prerequisites: a basic knowledge of manifolds (tangent spaces, vector fields)

Content: Let $M \equiv (M, g)$ be a connected Riemannian manifold with dimension $m$. Then $M$ canonically becomes a metric space via the distance function

$$d(x, y) := \inf \{ L(\gamma) \mid \gamma : [0, 1] \to M, \gamma(0) = x, \gamma(1) = y \},$$

where

$$L(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt$$

denotes the length of the (sufficiently smooth) curve $\gamma$. Thus, one gets the associated open balls $B(x, r)$. Moreover, the Riemannian metric $g$ on $M$ induces a measure $\mu$ on the Borel sets of $M$, given locally by

$$d\mu(x) = \sqrt{|g|} dx$$

and a fundamental question is how the underlying geometry affect the quantities $\mu(B(x, r))$. Such quantities play a fundamental role, e.g., in the context of heat kernel estimates on $M$ (via the so called Li-Yau heat kernel estimates).

The main goal of this seminar is to explain the following fundamental comparison estimates for these quantities, where given $\kappa \leq 0$, $r > 0$, the quantity $V_\kappa(r)$ denote the volume of a ball of radius $r$ in the Hyperbolic space of dimension $m$ with constant sectional curvature $\kappa$ (with the convention that $V_\kappa|_{\kappa=0}(r)$ is the volume of a Euclidean ball of dimension $m$ and radius $r$).

I) The Bishop-Günther estimate, which roughly states if the sectional curvatures of $M$ are bounded from above by $\kappa$, then $\mu(B(x, r))$ is controlled from below by $V_\kappa(r)$, for all $r$ smaller than the injectivity radius of $M$ at $x$.

II) The Bishop estimate, which states that if the Ricci curvature of $M$ is bounded from below by $\kappa \cdot (m - 1)$, then $\mu(B(x, r))$ is controlled from above by $V_\kappa(r)$, for all $r$.

III) The Gromov estimate, which states that if the Ricci curvature of $M$ is bounded from below by $\kappa \cdot (m - 1)$, then $\mu(B(x, r))/V_\kappa(r)$ is a decreasing function of $r$.

A rough table of contents is:

- Connections and exponential coordinates
- Riemannian manifolds as metric measure spaces
- Jacobi fields
- Conjugate and Cut loci
- The Bishop-Günther estimate
- The Bishop estimate
- The Gromov estimate