Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
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Sheet 11: due on Friday 11 January at 12:00 in Room 1.032.

For all $l, l' \in \mathbb{N}$ define

$$e^{(D_x, D_\zeta)} := F^{-1} e^{i(x,\zeta)} F : \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

and let $k \in \mathbb{R} \cup \{-\infty\}$. With some effort[1] one can show that $e^{(D_x, D_\zeta)}$ induces a map

$$e^{(D_x, D_\zeta)} : S^k_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})),$$

where S^k_c is understood to be the space of symbols having a compact support in the x-variable, and that

$$e^{(D_x, D_\zeta)} p \sim \sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\zeta p.$$

1 The formal adjoint [10 points]

Show that for all $p \in S^k_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ one has $\mathrm{Op}(p)^\dagger = \mathrm{Op}(q)$, where

$$q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$$

is given by $q = e^{(D_x, D_\zeta)} p^\dagger$.

2 Composition of compactly supported symbols [10 points]

Show that for all $r \in \mathbb{R} \cup \{-\infty\}$, $l'' \in \mathbb{N}$, the map

$$\rho : S^k_c(\Omega, \mathrm{Mat}_{l'\times l''}(\mathbb{C})) \times S^r_c(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \longrightarrow S^{k+r}_c(\Omega, \mathrm{Mat}_{l\times l''}(\mathbb{C}))$$

given by

$$\rho(p, q)(x, \zeta) := e^{(D_y, D_\zeta)} p(x, \zeta) q(y, \eta)|_{y=x, \eta=\zeta}$$

is well-defined, bilinear, and continuous.

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[1]cf. Section 6.5 in van den Ban / Crainic: *Analysis on manifolds*. Lecture notes available online.