Exercises in Global Analysis II

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Sheet 12: due on Friday 18 January at 12:00 in Room 1.032.

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1 Elliptic differential operators? [8 points]

Are the following differential operators elliptic or not? Prove your answer.

(a) The operator

$$\sum_{i,j=1}^{m} g^{ij}\partial_{x_i}\partial_{x_j} : \mathscr{D}(\Omega) \longrightarrow \mathscr{E}(\Omega),$$

where $(g^{ij})_{ij}$ is a symmetric positive semidefinite matrix.

(b) The operator

$$\partial_{x_{m+1}} + \sum_{i,j=1}^{m} g^{ij}\partial_{x_i}\partial_{x_j} : \mathscr{D}(\Omega\times\mathbb{R}) \longrightarrow \mathscr{E}(\Omega\times\mathbb{R}),$$

with g^{ij} as above.

(c) The operator

$$(1/2)\partial_x + \sqrt{-1}\partial_y : \mathscr{D}(\mathbb{R}^2) \longrightarrow \mathscr{D}(\mathbb{R}^2).$$

(d) The operator

$$\sum_{j=1}^{m}(i\partial_{x_j} - a_j)^2 : \mathscr{D}(\Omega) \longrightarrow \mathscr{E}(\Omega),$$

where $a \in \mathscr{E}(\Omega, \mathbb{R}^m)$.

2 A symbol estimate for ellipticity [6 points]

Show that a differential operator

$$P = \sum_{|\alpha|\le k} P_\alpha \partial^\alpha : \mathscr{D}(\Omega) \longrightarrow \mathscr{E}(\Omega)$$

is elliptic, if and only if for each $x \in \Omega$ there exists $C(x) > 0$ such that

$$1 + \left|\sum_{|\alpha|\le k} P_\alpha(x)\zeta^\alpha\right| \ge C(x)(1+|\zeta|^2)^{k/2}.$$

3 Sobolev spaces and diffeomorphisms [6 points]

Let $\Omega' \subset \mathbb{R}^m$ be another open subset, let $\phi : \Omega \to \Omega'$ be a diffeomorphism, and let $T\colon \Omega \to \mathrm{GL}(\mathbb{C}^l)$ be a smooth map. Show that for all $s \in \mathbb{N}$, $f' \in H^s_{\mathrm{loc}/c}(\Omega', \mathbb{C}^l)$ one has $T \cdot (f' \circ \phi) \in H^s_{\mathrm{loc}/c}(\Omega, \mathbb{C}^l)$ and that the induced maps

$$H^s_{\mathrm{loc}/c}(\Omega', \mathbb{C}^l) \longrightarrow H^s_{\mathrm{loc}/c}(\Omega, \mathbb{C}^l)$$

are isomorphisms of locally convex spaces.