# Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
Professor: Batu Güneysu
Assistant: Koen van den Dungen

**Sheet 6: due on Friday 23 November at 12:00 in Room 1.032.**

---

## 1 Absolutely continuous functions [5 points]

Let $I = (a, b)$ be an open interval, where $-\infty \leq a < b \leq \infty$. Consider a locally integrable function $f \in L^1_{\mathrm{loc}}(I, \mathbb{C}^l)$ and a point $x_0 \in I$, and let $F : I \to \mathbb{C}^l$ be the function given for all $x \in I$ by

$$F(x) = F(x_0) + \int_{x_0}^{x} f(y)dy.$$

Such functions $F$ are called locally *absolutely continuous*, and the space of locally absolutely continuous functions $I \to \mathbb{C}^l$ is denoted by $\mathrm{AC}_{\mathrm{loc}}(I, \mathbb{C}^l)$.
Show that $F$ is continuous, and that $\partial T_F = T_f$ in $\mathscr{D}'(I, \mathbb{C}^l)$, where (as usual) $T_g$ denotes the distribution in $\mathscr{D}'(I, \mathbb{C}^l)$ corresponding to a function $g \in L^1_{\mathrm{loc}}(I, \mathbb{C}^l)$.
**Remark:** in fact, much more is true: $F$ is differentiable almost everywhere and $F' = f$ almost everywhere (but this is not part of the exercise).

## 2 Sobolev spaces on an interval [5 points]

Let $I = (a, b)$ be an open interval, where $-\infty \leq a < b \leq \infty$.

(a) Consider a distribution $T \in \mathscr{D}'(I, \mathbb{C}^l)$ such that $\partial T = 0$. Show that $T$ is given by a constant vector in $\mathbb{C}^l$.

(b) Show that $f \in H^1(I, \mathbb{C}^l)$ if and only if $f$ is equal almost everywhere to a function $\tilde{f} \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{C}^l) \cap L^2(I, \mathbb{C}^l)$ such that $\partial T_{\tilde{f}} \in L^2(I, \mathbb{C}^l)$.

(c) For any $k \in \mathbb{N}_{\geq 1}$, show that $f \in H^k(I, \mathbb{C}^l)$ if and only if $f$ is equal almost everywhere to a function $\tilde{f} \in C^{k-1}(I, \mathbb{C}^l)$ such that $\tilde{f}^{(n)} \in L^2(I, \mathbb{C}^l)$ for all $0 \leq n \leq k-1$, $\tilde{f}^{(k-1)} \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{C}^l)$, and $\partial T_{\tilde{f}^{(k-1)}} \in L^2(I, \mathbb{C}^l)$.

*Hint:* you may use the Remark from Exercise 1.

## 3 Fourier convolution [5 points]

(a) Given Schwartz functions $f, g \in \mathscr{S}(\mathbb{R}^m)$, prove that

$$F(gf) = Fg * Ff.$$

(b) Given $g \in C_c^\infty(\mathbb{R}^m)$ and $f \in H^s(\mathbb{R}^m)$ for some $s \in \mathbb{R}$, prove that

$$F(gf) = Fg * Ff.$$

(c) Show that the Fourier transform of a compactly supported function $f \in H^s(\mathbb{R}^m)$ (for some $s \in \mathbb{R}$) is smooth.

## 4 Duality for Sobolev spaces [5 points]

We have seen in the lectures (Proposition 3.56.6) that the scalar product $\langle \cdot, \cdot \rangle_{L^2}$ on $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ extends continuously to an antidual pairing between $H^s(\mathbb{R}^m, \mathbb{C}^l)$ and $H^{-s}(\mathbb{R}^m, \mathbb{C}^l)$. Use this to prove that we have a canonical isomorphism of Banach spaces $H^s(\mathbb{R}^m, \mathbb{C}^l)' \simeq H^{-s}(\mathbb{R}^m, \mathbb{C}^l)$.