Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
Professor: Batu Güneysu
Assistant: Koen van den Dungen

UNIVERSITÄT BONN
Rheinische Friedrich-Wilhelms-Universität Bonn

Sheet 7: due on Friday 30 November at 12:00 in Room 1.032.

---

1 Differential operators on Sobolev spaces [5 points]

Let $\Omega \subset \mathbb{R}^m$ be an open subset. Prove that every partial differential operator $P = \sum_\alpha P_\alpha D^\alpha$ of order $\leq k$ with $P_\alpha \in C^\infty(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ gives continuous maps

$$P\colon H^s_{\mathrm{loc}}(\Omega, \mathbb{C}^l) \to H^{s-k}_{\mathrm{loc}}(\Omega, \mathbb{C}^{l'}), \qquad P\colon H^s_c(\Omega, \mathbb{C}^l) \to H^{s-k}_c(\Omega, \mathbb{C}^{l'}).$$

2 Proof of Theorem 3.62 [5 points]

Assume we are given a linear operator $T\colon L^2(\Omega, \mathbb{C}^l) \to L^2(\Omega, \mathbb{C}^l)$ for which there exist continuous linear operators $T_1, T_2\colon L^2(\Omega, \mathbb{C}^l) \to L^2(\Omega, \mathbb{C}^l)$ such that $T = T_1 T_2$ and numbers $s \in \mathbb{R}$, $k \in \mathbb{N}_{\geq 1}$ with $s > k + \frac{m}{2}$ and $T_j\big(L^2(\Omega, \mathbb{C}^l)\big) \subset H^s_{\mathrm{loc}}(\Omega, \mathbb{C}^l)$ for both $j = 1, 2$. Prove that one has $T\big(L^2(\Omega, \mathbb{C}^l)\big) \subset C^k(\Omega, \mathbb{C}^l)$ and that there exists a map $T(\cdot,\cdot) \in C^{k-1}(\Omega \times \Omega, \mathrm{Mat}_{l\times l}(\mathbb{C}))$ with

$$\int_\Omega |T(x,y)|^2 dy < \infty, \qquad Tf(x) = \int_\Omega T(x,y) f(y) dy,$$

for all $x \in \Omega$ and $f \in L^2(\Omega, \mathbb{C}^l)$.
Hint: In the lecture notes the proof is given for real-valued functions. The exercise is therefore to extend the proof to complex vector-valued functions.

3 Examples of symbols [5 points]

Consider the functions $a_j\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}$ given by

$$a_1(x,\xi) = \sum_{|\alpha|\leq k} P_\alpha(x) \xi^\alpha (1 + |\xi|^2)^{-s/2}, \qquad k \in \mathbb{N},\ s \in \mathbb{R},\ P_\alpha \in C^\infty(\mathbb{R}^m),$$

$$a_2(x,\xi) = \Big(1 + |\xi|^2 + \sum_{i,j=1}^m x^i x^j \xi_i \xi_j\Big)^{-1}.$$

Show that these functions define symbols $a_1 \in S^{k-s}(\mathbb{R}^m)$ and $a_2 \in S^{-2}(\mathbb{R}^m)$.
Remark: The symbol a_1 corresponds to a pseudo-differential operator $P(1+\Delta)^{-s/2}$, where P is a differential operator of order k and Δ is the Laplacian on $\mathbb{R}^m$. The symbol a_2 corresponds to a *parametrix* for the operator $1 + \Delta + \sum_{i,j=1}^m x^i x^j D_i D_j$.

4 The distribution kernel of a pseudo-differential operator [5 points]

Let $\Omega \subset \mathbb{R}^m$ be an open subset. Consider a symbol $p \in C^\infty_c(\Omega \times \mathbb{R}^m, \mathrm{Mat}_{l\times l'}(\mathbb{C})) \subset S^d(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ and the corresponding pseudo-differential operator $\Psi_p\colon C^\infty_c(\Omega, \mathbb{C}^l) \to$

$C^\infty(\Omega, \mathbb{C}^{l'})$. Let $F_2 p$ denote the Fourier transform of p with respect to the second variable. Consider the function $K_p \colon \Omega \times \Omega \to \mathrm{Mat}_{l \times l'}(\mathbb{C})$ given by

$$K_p(x, y) := F_2 p(x, y - x).$$

Prove that K_p satisfies the following equality for all $f \in C_c^\infty(\Omega, \mathbb{C}^l)$ and $g \in C_c^\infty(\Omega, \mathbb{C}^{l'})$:

$$\langle \Psi_p f, g \rangle = \langle K_p, g \otimes f \rangle.$$

Remark: The distribution K_p is called the *distribution kernel* for Ψ_p, and the above definition can be extended to all symbols $p \in S^d(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ (which we will see later). The existence (and uniqueness) of K_p follows also from the *Schwartz kernel theorem*, which states that in fact each continuous linear operator $C_c^\infty(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C})) \to \mathcal{D}'(\Omega, \mathrm{Mat}_{l \times l'}(\mathbb{C}))$ has a (uniquely determined) distribution kernel.