Exercises in Global Analysis II

University of Bonn, Winter Semester 2018-2019
Professor: Batu Güneysu
Assistant: Koen van den Dungen

UNIVERSITÄT BONN

Rheinische Friedrich-Wilhelms-Universität Bonn

Sheet 9: due on Friday 14 December at 12:00 in Room 1.032.

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1 Proof of Lemma 4.18 [8 points]

Assume $U \subset \Omega \times \Omega$ is an open neighbourhood of $\mathrm{diag}(\Omega)$. Then for every open covering $(V_i)_{i\in I}$ of Ω there exists a smooth partition of unity $(\phi_i)_{i\in I}$ subordinate to $(V_i)_{i\in I}$ such that for all $i, j \in I$ with $\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq \emptyset$ one has $\mathrm{supp}(\phi_i) \times \mathrm{supp}(\phi_j) \subset U$.
Hint: Show that there exists a refinement $(V'_j)_{j\in J}$ of $(V_i)_{i\in I}$ which is locally finite such that for all $A, B \in (V'_j)_{j\in J}$ with $A \cap B$ nonempty one has $A \times B \subset U$.

2 Asymptotic expansions of symbols [6 points]

Given a countable set N, a function $N \to \mathbb{R}$, $n \mapsto k_n$, with $k_n \to -\infty$ as $n \to \infty$[1], $k \in \mathbb{R}$, a symbol $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, and for each $n \in N$ a symbol $p_n \in S^{k_n}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$. Then one writes

$$p \sim \sum_{n\in N} p_n,$$

if for all $k' \in \mathbb{R}$ there exists a finite set $F_0 \subset N$ such that for all finite subsets $F \subset N$ with $F_0 \subset F$ one has

$$p - \sum_{n\in F} p_n \in S^{k'}(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C})).$$

One then calls $\sum_{n\in N} p_n$ an *asymptotic expansion* of p.
Now let $k \in \mathbb{R}$, $p \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$, $\phi \in \mathscr{D}(\Omega)$, and define $q \in S^k(\Omega, \mathrm{Mat}_{l\times l'}(\mathbb{C}))$ by

$$q(x,\zeta) := \int_{\mathbb{R}^m} e^{i(x,y)} p(x, y+\zeta) F\phi(y)dy.$$

Show that

$$q \sim \sum_{\alpha\in\mathbb{N}^m} D^\alpha_x \phi \partial^\alpha_y p.$$

Hint: Use the multidimensional Taylor formula with remainder to rewrite $p(x, y + \zeta)$.

3 The support of a differential operator [6 points]

Prove that the Schwartz kernel K_P of a *differential* operator $P \in \Psi^k(\Omega, \mathbb{C}^l, \mathbb{C}^{l'})$ is supported on $\mathrm{diag}(\Omega) \subset \Omega \times \Omega$. Consequently, conclude that every differential operator is properly supported.

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[1]that is, for all $r \in \mathbb{R}$ there exists a finite set $F \subset N$ such that for all $n \in N \setminus F$ one has $k_n < r$.