Solution hints to Sheet 6:

1. If $h_+ \in W^{1,2}_0(M)$, then one has $h \leq v := h_+$.
Conversely, suppose that there exists $v \in W^{1,2}_0(M)$ with $h \leq v$. 
Case $v \in C_c^\infty(M)$. Pick $\phi \in C_c^\infty(M)$ with $0 \leq \phi \leq 1$ on $M$ and $\phi = 1$ on supp$(v)$. Then using $h \leq v$ one easily checks

$$h_+ = ((1 - \phi)v + \phi h)_+.$$

Since $\phi h$ is a compactly supported element of $W^{1,2}(M)$, one has $\phi h \in W^{1,2}_0(M)$, and so

$$(1 - \phi)v + \phi h \in W^{1,2}(M),$$

and thus by Example 3.6 one has

$$h_+ = ((1 - \phi)v + \phi h)_+ \in W^{1,2}_0(M).$$

Case $v \in W^{1,2}_0(M)$: Pick a sequence $v_n \in C_c^\infty(M)$ with $v_n \to v$ in $W^{1,2}(M)$. Then, because $h \leq v$, we have

$$h_n := h + (v_n - v) \leq v_n$$

and so by the previous case $(h_n)_+ \in W^{1,2}_0(M)$ for all $n$. Since $h_n \to h$ in $W^{1,2}(M)$, Example 3.6 gives $(h_n)_+ \to h_+$ in $W^{1,2}(M)$ and $W^{1,2}_0(M) \subset W^{1,2}(M)$ is a closed subspace, and we end up with $h_+ \in W^{1,2}_0(M)$.

2. The map

$$\varphi : H^m \to B_e^{\mathbb{R}^m}(0,1) = \{y \in \mathbb{R}^m : |y| < 1\}, \quad (x', x^{m+1}) \mapsto \frac{x'}{x^{m+1} + 1}, \quad (1)$$

where

$$|y|^2 := \sum_{i=1}^m y_i^2,$$

is a global chart with

$$\varphi^{-1} : B_e^{\mathbb{R}^m}(0,1) \to H^m, \quad y \mapsto \left(\frac{2y}{1 - |y|^2}, \frac{1 + |y|^2}{1 - |y|^2}\right). \quad (2)$$
Since for all \((x', x^{m+1}) \in H^m\) one has
\[(x^{m+1})^2 - (x')^2 = 1\]
we get (with the usual abuse of notation denoting the map of the charts and the points with the same symbol)
\[x^{m+1} dx^{m+1} = \sum_{i=1}^{m} (x')^i d(x')^i\]
and (1) implies that for all \(i = 1, \ldots, m\),
\[dy^i = \frac{(1 + x^{m+1}) d(x')^i - (x')^i dx^{m+1}}{(1 + x^{m+1})^2}.
\]
Using the latter two formulae one calculates
\[\sum_{i=1}^{m} dy^i \otimes dy^i = (1 + x^{m+1})^{-2} \sum_{i=1}^{m} d(x')^i \otimes d(x')^i - dx^{m+1} \otimes dx^{m+1},\]
and so
\[\sum_{i=1}^{m} d(x')^i \otimes d(x')^i - dx^{m+1} \otimes dx^{m+1} = (1 + x^{m+1})^2 \sum_{i=1}^{m} dy^i \otimes dy^i.\]
Since by (2)
\[1 + x^{m+1} = 2/(1 - |y|^2),\]
we end up with the following formula in the above chart (with \(g_e^{R^m}\) the Euclidean metric on \(R^m\)):
\[g_{\text{Mink}} = \frac{4}{1 - |\cdot|^2} g_e^{R^m}\] on \(H^m \cong B_e^{R^m}.\) (3)
This formula shows that \(g_{\text{Mink}}\) is positive definite on \(H^m\).
To see that \(\mathbb{H}^m := (H^m, g_{\text{Mink}})\) is complete, note that a (connected) Riemannian manifold \(M\) is complete, if and only if for every piecewise smooth curve \(\gamma : [0, \infty) \rightarrow M\) which eventually leaves every compact subset of \(M\) one has
\[\int_0^\infty |\dot{\gamma}(t)| dt = \infty.\]
The latter condition can be checked easily on \(\mathbb{H}^m\) using (3), noting that a continuous curve \(\gamma : [0, \infty) \rightarrow H^m\) which eventually leaves every compact subset of \(H^m\), if and only if \(\gamma\) eventually touches a point on \(\partial B_e^{R^m}\) and remains then there for all times.