Solution hints to Sheet 7:

1. There will be no danger in denoting $h|_{U_n}$ again with $h$ for functions $h$ on $M$, and conversely, the extension of a function $h$ from a subset of $M$ to the whole of $M$ will also be denoted by $h$ again.

Step 1: For all $\alpha > 0$ one has

$$u_n := (H^{U_n} + \alpha)^{-1} f \to (H + \alpha)^{-1} f$$

in $W^{1,2}(M)$.

Proof of step 1: All scalar products and norms below are in the sense of $M$, while an index $n$ in these data will mean that we work on $U_n$. We have $0 \leq u_n \leq (H + \alpha)^{-1} f$, and $u_n \not\to u \mu$-a.e. to a function $u$ with $0 \leq u(H + \alpha)^{-1} f$. A posteriori, $u \in L^2(M)$ and $u_n \to u$ in $L^2(M)$ by dominated convergence, in particular, $u_n$ is Cauchy in $L^2(M)$. It remains to show $u = (H + \alpha)^{-1} f$.

We have $u_n \in W^{1,2}_0(U_k) \subset W^{1,2}_0(M)$ for all $k > n$. For all $n$ and all $\phi \in W^\infty_0(U_n)$ we have

$$\langle du_n, d\phi \rangle + \alpha \langle u_n, \phi \rangle = \langle f, \phi \rangle,$$

and so

$$\langle du_n, du_n \rangle + \alpha \langle u_n, u_n \rangle = \langle f, u_n \rangle.$$

If $k > n$ then $\phi = u_k - 2u_n \in W^{1,2}(U_k)$ and so using (1) with $n$ replaced by $k$ we get

$$\langle du_k, d(u_k - 2u_n) \rangle + \alpha \langle u_k, u_k - 2u_n \rangle = \langle f, u_k - 2u_n \rangle.$$

The sum of the last two formula gives

$$\|du_k\|^2 + \|du_n\|^2 - 2 \langle du_k, du_n \rangle + \alpha \left( \|u_k\|^2 + \|u_n\|^2 - 2 \langle u_k, u_n \rangle \right)$$

$$= \langle f, u_k - u_n \rangle,$$

so

$$\|d(u_k - u_n)\|^2 + \alpha \|u_k - u_n\|^2 = \langle f, u_k - u_n \rangle \leq \|f\| \|u_k - u_n\|.$$

Since $u_n$ is Cauchy in $L^2(M)$, the last inequality shows that $du_n$ is Cauchy in $\Omega^1_{L^2}(M)$. Thus $u_n$ is Cauchy in $W^{1,2}_0(M)$ and its limit is $u$, and so
\( u \in W^{1,2}(M) \).

Let \( \phi \in C_c^{\infty}(M) \). For all \( n \) large enough such that the support of \( \phi \) sits in \( U_n \) one has (1), and so taking \( n \to \infty \) we obtain
\[
\langle du, d\phi \rangle + \alpha \langle u, \phi \rangle = \langle f, \phi \rangle,
\]
which extends to arbitrary \( \phi \in W^{1,2}_0(M) \). Examining the description of the domain of definition of the nonnegative self-adjoint operator associated to a closed nonnegative form, this formula shows that \( u \in \text{Dom}(H) = \text{Dom}(H + \alpha) \) and \( (H + \alpha)u = f \), and so \( u = (H + \alpha)^{-1}f \). This completes the proof of step 1.

Step 2: For all \( t > 0 \) one has \( P_t U_n f \to P_t f \) \( \mu \)-a.e.

Proof of step 2: We have as above we have \( P_t U_n f \not\to u_t \) \( \mu \)-a.e. for some \( 0 \leq u_t \in L^2(M) \), and a posteriori \( P_t U_n f \to u_t \) in \( L^2(M) \). It remains to show \( u_t = P_t f \). For all \( 0 \leq \phi \in C_0^{\infty}(M) \). For all large enough \( n \) such that the support of \( \phi \) sits in \( U_n \). Taking Laplace transforms, we get
\[
\langle (H U_n + \alpha)^{-1} f, \phi \rangle = \int_0^\infty e^{-\alpha s} \langle P_s U_n f, \phi \rangle ds \to \int_0^\infty e^{-\alpha s} \langle u_s f, \phi \rangle ds
\]
from monotone convergence. On the other hand, by step 1 we have
\[
\langle (H U_n + \alpha)^{-1} f, \phi \rangle \to \langle (H + \alpha)^{-1} f, \phi \rangle = \int_0^\infty e^{-\alpha s} \langle P_s f, \phi \rangle ds,
\]
and so
\[
\int_0^\infty e^{-\alpha s} \langle u_s f, \phi \rangle ds = \int_0^\infty e^{-\alpha s} \langle P_s f, \phi \rangle ds.
\]
Since for all \( s > 0 \),
\[
\langle u_s, \phi \rangle \leq \langle P_s f, \phi \rangle,
\]
the previous formula implies
\[
\langle u_s, \phi \rangle = \langle P_s f, \phi \rangle,
\]
for almost all \( s > 0 \). The RHS is continuous in \( s \), and it remains to remains to show that the LHS is continuous in \( s \), too. For all \( n \) we have
\[
\left| \langle (d/ds) P_s U_n f, \phi \rangle \right| \leq \|f\| \|\Delta \phi\|,
\]
which is easily seen using that \( P_s U_n f \) solves the heat equation with respect to \( H U_n \), integrating by parts and using heat semigroups are contractive. This shows that \( s \mapsto P_s U_n f, n \in \mathbb{N} \), is uniformly Lipschitz, thus the pointwise limit of this sequence of function on \((0, \infty)\), namely, \( s \mapsto \langle u_s, \phi \rangle \) is Lipschitz.
2. Fix $t \in I$. For all $x \in U$, $\epsilon > 0$ we can find $\theta \in (0, 1)$, such that

$$ \frac{\zeta(t + \epsilon, x) - \zeta(t, x)}{\epsilon} = \partial_t \zeta(t + \theta \epsilon, x), $$

but

$$ \sup_{x \in U} \left| \frac{\zeta(t + \epsilon, x) - \zeta(t, x)}{\epsilon} - \partial_t \zeta(t, x) \right| = \sup_{x \in U} |\partial_t \zeta(t + \theta \epsilon, x) - \partial_t \zeta(t, x)| \to 0 $$

as $\epsilon \to 0^+$, as $\partial_t \zeta$ is uniformly continuous in $[t/2,(3t)/2] \times U$. This is the claimed strong differentiability.