Solution hints to Sheet 9:

2. Let $C$ be the doubling constant. Assume first $S = 2s$. Since $M$ is connected, there exists $y \in M$ with $\varrho(x, y) = (3r)/2$ and so

$$B(y, s/2) \subset B(x, 2s) \setminus B(x, s),$$

and so

$$\mu(x, 2s) \geq \mu(x, s) + \mu(y, s/2),$$

and

$$\frac{\mu(x, s)}{\mu(y, s/2)} \leq \frac{\mu(y, 4s)}{\mu(y, s/2)} \leq C^3.$$

We arrive at

$$\mu(x, 2s) \geq (1 + C^{-3})\mu(x, s).$$

Iterating this (as in the proof of Li-Yau) and setting $n := \log_2(1 + 1/C^3)$, we get, for all $s \leq S$ and $x$,

$$\frac{\mu(x, S)}{\mu(x, s)} \geq (1 + 1/C^3)(S/s)^n$$

3. Assume $e_j$, $j \in \mathbb{N}$, is an ONB of the separable Hilbert space $L^2(M)$. Then we have

$$\text{tr}(P_t) = \sum_j \langle Pe_j, e_j \rangle = \sum_j \|Pe_j\|^2 = \int_M p(t/2, x, y)^2 d\mu(y)$$

$$= \int_M p(t, x, x) d\mu(x).$$