Exercises in Global Analysis II
University of Bonn, Winter Semester 2018-2019
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Sheet 2: due on Friday 26 October at 12:00 in Room 1.032.

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1 Subspaces and products [5 points]

(a) If X is a locally convex space and $A \subset X$ is a linear subspace (equipped with the induced topology), then A is again a locally convex space.

(b) If X is a Fréchet space and $A \subset X$ is a closed linear subspace, then A is again a Fréchet space.

(c) Let X and Y be locally convex spaces with the topologies induced by the seminorms $(p_i)_{i\in I}$ on X and $(q_j)_{j\in J}$ on Y. Then the product space $X \times Y$ is again a locally convex space, and the product topology is equal to the topology induced by the seminorms $(p_i \times q_j)_{(i,j)\in I\times J}$ defined for $(x, y) \in X \times Y$ by

$$p_i \times q_j(x, y) := p_i(x) + q_j(y).$$

Moreover, if X and Y are Fréchet spaces, then $X \times Y$ is also a Fréchet space.

2 Open mapping theorem [5 points]

Let X and Y be Fréchet spaces, and let $T \in \mathcal{L}(X, Y)$ be surjective. Prove that T is open.
Hint: Use the following consequence of the Baire category theorem:
Let X be a nonempty topological space whose topology is induced by a complete metric. If $\cup_{n\in\mathbb{N}} A_n$ is a countable covering of X with closed subsets, then there exists an $l \in \mathbb{N}$ such that A_l has a nonempty interior.

3 Schwartz space [5 points]

Prove that the Schwartz space $\mathscr{S}(\mathbb{R}^m, \mathbb{C}^l)$ is complete with respect to the topology induced by the seminorms

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right|,$$

where $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^l)$ and $\alpha, \beta \in \mathbb{N}^n$.

4 Weak versus strong differentiability [5 points]

Let X be a Banach space, and consider a map $f : \mathbb{R} \to X$.

(a) Suppose that for each $\phi \in X^*$ the map $t \mapsto \phi\big[f(t)\big]$ is continuously differentiable. Prove that f is continuous.

(b) Suppose that for each $\phi \in X^*$ the map $t \mapsto \phi\big[f(t)\big]$ is k times continuously differentiable for some integer $k > 1$. Prove that f is $k - 1$ times differentiable.