Minimal Supersolutions of Convex BSDEs*

Samuel Drapeau† Gregor Heyne ‡ Michael Kupper §

October 25, 2012

We study the nonlinear operator of mapping the terminal value ξ to the corresponding minimal supersolution of a backward stochastic differential equation with the generator being monotone in y, convex in z, jointly lower semicontinuous, and bounded below by an affine function of the control variable z. We show existence, uniqueness, monotone convergence, Fatou's Lemma and lower semicontinuity of this operator. We provide a comparison principle for minimal supersolutions of BSDEs.

Keywords: Supersolutions of Backward Stochastic Differential Equations; Non-Linear Expectations; Supermartingales

1 Introduction

On a filtered probability space, where the filtration is generated by a d-dimensional Brownian motion W, we consider the process $\hat{\mathcal{E}}^g(\xi)$ given by

$$\hat{\mathcal{E}}^g_t(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0_t : (Y,Z) \in \mathcal{A}(\xi,g)\right\}, \quad t \in [0,T],$$

where $\mathcal{A}(\xi,g)$ is the set of all pairs of càdlàg *value processes* Y and *control processes* Z such that

$$Y_s - \int_s^t g_u(Y_u,Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \tag{1.1}$$

for all $0 \leq s \leq t \leq T$. Here the *terminal condition* ξ is a random variable, the *generator* g a measurable function of (y,z) and the pair (Y,Z) is a *supersolution* of the *backward stochastic differential equation (BSDE)* (1.1).

The main objective of this paper is to state conditions which guarantee that there exists a unique minimal supersolution. More precisely, we show that the process $\mathcal{E}^g(\xi) = \lim_{s\downarrow\cdot, s\in\mathbb{Q}} \hat{\mathcal{E}}^g_s(\xi)$ is a modification of $\hat{\mathcal{E}}^g(\xi)$ and equals the value process of the unique minimal supersolution, that is, there exists a unique

*We thank Patrick Cheridito, Hans Föllmer, Ramon Van Handel, Ulrich Horst and Reinhard Schmidt for helpful comments and fruitful discussions. We thank an anonymous referee for careful reading and helpful comments.

†drapeau@math.hu-berlin.de; Humboldt University Berlin, Unter den Linden 6, D-10099 Berlin, Germany. Financial support from MATHEON project E.11 is gratefully acknowledged.

‡heyne@math.hu-berlin.de; Humboldt University Berlin, Unter den Linden 6, D-10099 Berlin, Germany.

§kupper@math.hu-berlin.de; Humboldt University Berlin, Unter den Linden 6, D-10099 Berlin, Germany.

control process $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. The existence theorem immediately yields a comparison theorem for minimal supersolutions. We also study the stability of the minimal supersolution with respect to the terminal condition and the generator. We show that the mapping $\xi \mapsto \hat{\mathcal{E}}^g_0(\xi)$ is a nonlinear expectation, fulfills a montone convergence theorem and Fatou's Lemma on the same domain as the expectation operator $E[\cdot]$, and consequently is L^1-lower semicontinuous.

Nonlinear expectations have been a prominent topic in mathematical economics since Allais famous paradox, see Föllmer and Schied [21, Section 2.2]. Typical examples are the monetary risk measures introduced by Artzner et al. [2] and Föllmer and Schied [20], Peng's g and G-expectations, see [29, 31, 32], the variational preferences by Maccheroni et al. [28], and the recursive utilities by Duffie and Epstein [15]. Especially the g-expectation, which is defined as the initial value of the solution of a BSDE, is closely related to $\mathcal{E}^g_0(\cdot)$, since each pair (Y, Z) that solves the BSDE corresponding to (1.1) is also a supersolution and hence an element of $\mathcal{A}(\xi, g)$. The concept of a supersolution of a BSDE appears already in El Karoui et al. [18, Section 2.2]. For further references see Peng [30], who derives monotonic limit theorems for supersolutions of BSDEs and proves the existence of a minimal constrained supersolution.

Our first contribution is to provide a setting where we relax the usual Lipschitz requirements for the generator g. Namely, we suppose that g is convex with respect to z, monotone in y, jointly lower semicontinuous, and bounded below by an affine function of the control variable z. To see in an intuitive way the role these assumptions play in deriving the existence and uniqueness of a control process $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, let us suppose for the moment that g is positive. Given an adequately good space of control processes, the value process of each supersolution and the process $\hat{\mathcal{E}}^g(\xi)$ are in fact supermartingales. By suitable pasting, we may now construct a decreasing sequence (Y^n) of supersolutions, whose pointwise limit is again a supermartingale and equal to $\hat{\mathcal{E}}^g(\xi)$ on all dyadic rationals. Since the generator g is positive, it can be shown that $\mathcal{E}^g(\xi)$ lies below $\hat{\mathcal{E}}^g(\xi)$, P-almost surely, at any time. This suggests to consider the càdlàg supermartingale $\mathcal{E}^g(\xi)$ as a candidate for the value process of the minimal supersolution. However, it is not clear a priori that the sequence (Y^n) converges to $\mathcal{E}^g(\xi)$ in some suitable sense. Yet, taking into account the additional supermartingale structure, in particular the Doob-Meyer decomposition, it follows that (Y^n) converges $P \otimes dt$-almost surely to $\mathcal{E}^g(\xi)$. It remains to obtain a unique control process $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. To that end, we prove that, for monotone sequences of supersolutions, a positive generator yields, after suitable stopping, a uniform L^1-bound for the sequence of supremum processes of the associated sequence of stochastic integrals. This, along with a result by Delbaen and Schachermayer [10], and standard compactness arguments and diagonalization techniques yield the candidate control process $\hat{Z}$ as the limit of a sequence of convex combinations. Now, joint lower semicontinuity of g, positivity, and convexity in z allow us to use Fatou's Lemma to verify that the candidate processes $(\mathcal{E}^g(\xi), \hat{Z})$ are a supersolution of the BSDE. Thus, $\mathcal{E}^g(\xi)$ is in fact the value process of the minimal supersolution and a modification of $\hat{\mathcal{E}}^g(\xi)$. Finally, the uniqueness of $\hat{Z}$ follows from the uniqueness of the Doob-Meyer decomposition of the càdlàg supermartingale $\mathcal{E}^g(\xi)$.

Let us give further reference of related assumptions and methods in the existing literature. Delbaen et al. [12] consider superquadratic BSDEs with generators that are positive and convex in z but do not depend on y. However, their principal aim and method differ from ours. Indeed, they primarily study the well-posedness of superquadratic BSDEs by establishing a dual link between cash additive time-consistent dynamic utility functions and supersolutions of BSDEs. To view supersolutions as supermartingales is one of the key ideas in our approach and we make ample use of the rich structure supermartingales provide. The classical limit theorem of supermartingales has been used by El Karoui and Quenez [16] in the theory of BSDEs, when studing the problem of option pricing in incomplete financial markets. However, the analysis is done via dual formulations and only for linear generators that do not depend on y. The construction of solutions of BSDEs by monotone approximations is also a classical tool, see for example Kobylanski [27] for quadratic generators and Briand and Hu [6] for generators that are in addition convex in z. The application of compactness theorems such as Delbaen and Schachermayer [10, Lemma A1.1], or Delbaen and Schachermayer [11, Theorem A], in order to derive existence of BSDEs seems to be new to the best of our knowledge. Often existence proofs rely on a priori estimates combined with a fixed point theorem, see for example [18], or on constructing Cauchy sequences in complete spaces, see for example Briand and Confortola [5] or Ankirchner et al. [1]. Recent exceptions are Réveillac [35] and Heyne et al. [25] who use a compactness result given in Barlow and Protter [3]. As already mentioned, Peng [30] studies the existence and uniqueness of minimal supersolutions. However, he assumes a Lipschitz generator, a square integrable terminal condition, and employs a very different approach. It is based on a monotonic limit theorem, [30, Theorem 2.4], the penalization method introduced in El Karoui et al. [17], and it leads to increasing sequences of supersolutions. Parallel to us, Cheridito and Stadje [7] have investigated existence and stability of supersolutions of BSDEs. They consider generators that are convex in z and Lipschitz in y. However, their setting and methods are quite different from ours. Namely, they approximate by discrete time BSDEs and work with terminal conditions that are bounded lower semicontinuous functions of the Brownian motion. An interesting equivalence between the minimal supersolution and the solution of a reflected BSDEs is given in Peng and Xu [33]. In [25] the authors show the existence of the minimal supersolution for generators that are lower semicontinuous, monotone in the value variable, bounded below by an affine function of the control variable, and which satisfy a specific normalization condition. Finally, given our local L^1-bounds, the compactness underlying the construction of the candidate control process is a special case of results obtained by Delbaen and Schachermayer [11].

Our second contribution is to allow for *local supersolutions*, that is for supersolutions (Y, Z), where the stochastic integral of Z is only a local martingale. However, in order to avoid so-called "doubling strategies", present even for the simplest generator $g \equiv 0$, see Dudley [14] or Harrison and Pliska [23, Section 6.1], we require in addition that $\int ZdW$ is a supermartingale. This specification interacts nicely with a positive generator and happens to be particularly adequate to establish stability properties of the minimal supersolution with respect to the terminal condition or the generator. In particular, it allows us to formulate theorems such as montone convergence and Fatou's lemma for the non-linear operator $\hat{\mathcal{E}}^g_0(\cdot)$ on the same domain as the standard expectation $E[\cdot]$ and to obtain its L^1-lower semicontinuity. Moreover, under some additional integrability on the terminal condition, our approach also allows to derive existence results with control processes, whose stochastic integrals belong to $\mathcal{H}^1$.

Dropping the positivity assumption, the value and control processes of our supersolutions are supermartingales under another measure closely linked to the generator g. In fact, for a positive generator we have supermartingales with respect to the initial probability measure P, while for a non-positive generator, which is bounded below by an affine function of the control variable, we consider supermartingales under the measure given by the corresponding Girsanov transform.

The paper is organized as follows. In Section 2 we fix our notations and the setting. We define minimal supersolutions, and introduce our main conditions and structural properties of $\hat{\mathcal{E}}^g(\xi)$ in Section 3. Finally in Section 4 we state and prove our main results - existence and stability theorems.

2 Setting and Notations

We consider a fixed time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and fulfills the usual conditions. We further assume that $\mathcal{F} = \mathcal{F}_T$. The set of $\mathcal{F}$-measurable and $\mathcal{F}_t$-measurable random variables is denoted by L^0 and L^0_t, respectively, where random variables are identified in the P-almost sure sense. The sets L^p and L^p_t denote the set of random variables in L^0 and L^0_t, respectively, with finite p-norm, for $p \in [1, +\infty]$. Throughout this work, inequalities and strict inequalities between any two random variables or processes X^1, X^2 are understood in the P-almost sure or in the $P \otimes dt$-almost sure sense, respectively, that is, $X^1 \leq (<) X^2$ is equivalent to $P\left[X^1 \leq (<) X^2\right] = 1$ or $P \otimes dt\left[X^1 \leq (<) X^2\right] = 1$, respectively. Given a process X and $t \in [0,T]$ we denote $X^*_t := \sup_{s\in[0,t]} |X_s|$. We denote by $\mathcal{T}$ the set of stopping times with values in $[0,T]$ and hereby call an increasing sequence of stopping times (τ^n), such that $P[\bigcup_n \{\tau^n = T\}] = 1$, a *localising sequence of stopping times*. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of all càdlàg progressively measurable processes Y with values in $\mathbb{R}$. For $p \in [1, +\infty[$, we further denote by $\mathcal{L}^p := \mathcal{L}^p(W)$ the set of progressively measurable processes Z with values in $\mathbb{R}^{1\times d}$, such that $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T Z_s Z_s^\top ds)^{p/2}]^{1/p} < +\infty$. For any $Z \in \mathcal{L}^p$, the stochastic integral $(\int_0^t Z_s dW_s)_{t\in[0,T]}$ is well defined, see [34], and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. For the $\mathcal{L}^p$-norm, the set $\mathcal{L}^p$ is a Banach space, see [34]. We further denote by $\mathcal{L} := \mathcal{L}(W)$ the set of progressively measurable processes with values in $\mathbb{R}^{1\times d}$, such that there exists a localising sequence of stopping times (τ^n) with $Z1_{[0,\tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. Here again, the stochastic integral $\int ZdW$ is well defined and is a continuous local martingale.

For adequate integrands a, Z, we generically write $\int a ds$ or $\int ZdW$ for the respective integral processes $(\int_0^t a_s ds)_{t\in[0,T]}$ and $(\int_0^t Z_s dW_s)_{t\in[0,T]}$. Finally, given a sequence (x_n) in some convex set, we say that a sequence (y_n) is in the *asymptotic convex hull* of (x_n), if $y_n \in conv\{x_n, x_{n+1}, \ldots\}$, for all n.

A *generator* is a jointly measurable function g from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$ to $\mathbb{R} \cup \{+\infty\}$ where $\Omega \times [0,T]$ is endowed with the progressive σ-field.

3 Minimal Supersolutions of BSDEs

3.1 Definitions

Given a generator g, and a *terminal condition* $\xi \in L^0$, a pair $(Y,Z) \in \mathcal{S} \times \mathcal{L}$ is a *supersolution* of a BSDE, if, for all $s, t \in [0,T]$, with $s \leq t$, it holds

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi. \tag{3.1}$$

For such a supersolution (Y,Z), we call Y the *value process* and Z its *control process*. Due to the càdlàg property, Relation (3.1) holds for all stopping times $0 \leq \sigma \leq \tau \leq T$, in place of s and t, respectively. Note that the formulation in (3.1) is equivalent to the existence of a càdlàg increasing process K, with $K_0 = 0$, such that

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u)du + (K_T - K_t) - \int_t^T Z_u dW_u, \qquad t \in [0,T]. \tag{3.2}$$

Although the notation in (3.2) is standard in the literature concering supersolutions of BSDEs, see for example [18, 30], we will keep with (3.1) since the proofs of our main results exploit this structure. We consider only those supersolutions $(Y,Z) \in \mathcal{S} \times \mathcal{L}$ of a BSDE where Z is *admissible*, that is, where the continuous local martingale $\int ZdW$ is a supermartingale. We are then interested in the set

$$\mathcal{A}(\xi, g) = \{(Y,Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is admissible and (3.1) holds}\} \tag{3.3}$$

and the process

$$\hat{\mathcal{E}}^g_t(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0_t : (Y,Z) \in \mathcal{A}(\xi, g)\right\}, \quad t \in [0,T]. \tag{3.4}$$

By $\hat{\mathcal{E}}^g$ we mean the functional mapping terminal conditions $\xi \in L^0$ to the process $\hat{\mathcal{E}}^g(\xi)$. If necessary, we write $\mathcal{A}_T(\xi, g)$ and $\hat{\mathcal{E}}^g_{\cdot,T}(\xi)$ for $\mathcal{A}(\xi, g)$ and $\hat{\mathcal{E}}^g(\xi)$, respectively, to indicate their dependence on the time horizon. Note that the essential infima in (3.4) can be taken over those $(Y,Z) \in \mathcal{A}(\xi, g)$, where $Y_T = \xi$. A pair (Y,Z) is called a *minimal supersolution*, if $(Y,Z) \in \mathcal{A}(\xi, g)$, and if for any other supersolution $(Y', Z') \in \mathcal{A}(\xi, g)$, holds $Y_t \leq Y'_t$, for all $t \in [0,T]$.

3.2 General Properties of $\mathcal{A}(\cdot, g)$ and $\mathcal{E}^g$

In this section we collect various statements regarding the properties of $\mathcal{A}(\cdot, g)$ and $\hat{\mathcal{E}}^g$. The first lemma ensures that the set of admissible control processes is stable under pasting and that we may concatenate elements of $\mathcal{A}(\xi, g)$ along stopping times and partitions of our probability space.

Lemma 3.1. *Fix a generator g, a terminal condition $\xi \in L^0$, a stopping time $\sigma \in \mathcal{T}$, and $(B^n) \subset \mathcal{F}_\sigma$ a partition of Ω.*

1. *Let $(Z^n) \subset \mathcal{L}$ be admissible. Then $\bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n\geq 1} Z^n 1_{B_n} 1_{]\sigma,T]}$ is admissible.*
2. *Let $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$ such that $Y^1_\sigma 1_{B^n} \geq Y^n_\sigma 1_{B^n}$, for all $n \in \mathbb{N}$. Then $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$, where*

$$\bar{Y} = Y^1 1_{[0,\sigma[} + \sum_{n\geq 1} Y^n 1_{B^n} 1_{[\sigma,T]} \quad \textit{and} \quad \bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n\geq 1} Z^n 1_{B^n} 1_{]\sigma,T]}. \tag{3.5}$$

Proof. 1. Let M^n and $\bar{M}$ denote the stochastic integrals of the Z^n and $\bar{Z}$, respectively. It follows from $(Z^n) \subset \mathcal{L}$ and from (B_n) being a partition that $\bar{Z} \in \mathcal{L}$ and that $\int_{s\vee\sigma}^{t\vee\sigma} \bar{Z}_u dW_u = \sum 1_{B_n} \int_{s\vee\sigma}^{t\vee\sigma} Z^n_u dW_u$. Now observe that the admissibility of all Z^n yields

$$E\left[\bar{M}_t - \bar{M}_s \,|\, \mathcal{F}_s\right] = E\left[M^1_{(t\wedge\sigma)\vee s} - M^1_s \,|\, \mathcal{F}_s\right] + E\left[\sum_{n\geq 1} 1_{B_n} E\left[M^n_{t\vee\sigma} - M^n_{s\vee\sigma} \,|\, \mathcal{F}_{s\vee\sigma}\right] |\, \mathcal{F}_s\right] \leq 0,$$

for $0 \leq s \leq t \leq T$.

2. $\bar{Z}$ is admissible by Item 1. Since $Y^1_\sigma 1_{B^n} \geq Y^n_\sigma 1_{B^n}$, for all $n \in \mathbb{N}$, it follows on the set $\{s < \sigma \leq t\}$ that

$$\begin{aligned}
Y^1_s - \int_s^\sigma g_u(Y^1_u, Z^1_u)du + \int_s^\sigma Z^1_u dW_u - \int_\sigma^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_\sigma^t \bar{Z}_u dW_u \\
\geq Y^1_\sigma - \sum_{n\geq 1} 1_{B^n} \left(\int_\sigma^t g_u(Y^n_u, Z^n_u)du - \int_\sigma^t Z^n_u dW_u\right) \\
\geq \sum_{n\geq 1} 1_{B^n} \left(Y^n_\sigma - \int_\sigma^t g_u(Y^n_u, Z^n_u)du + \int_\sigma^t Z^n_u dW_u\right) \geq \sum_{n\geq 1} 1_{B^n} Y^n_t.
\end{aligned}$$

Hence,

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u \geq 1_{\{\sigma > t\}} Y_t^1 + \sum_{n \geq 1} 1_{B^n} \left(1_{\{\sigma \leq s\}} Y_t^n + 1_{\{s < \sigma \leq t\}} Y_t^n\right) = \bar{Y}_t$$

and thus $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$. □

For convenience, a generator is said to be

(POS) positive, if $g\,(y, z) \geq 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

(INC) increasing, if $g\,(y, z) \geq g\,(y', z)$, for all $y, y' \in \mathbb{R}$ with $y \geq y'$, and all $z \in \mathbb{R}^{1 \times d}$.

(DEC) decreasing, if $g\,(y, z) \leq g\,(y', z)$, for all $y, y' \in \mathbb{R}$ with $y \geq y'$, and all $z \in \mathbb{R}^{1 \times d}$.

The following Lemma states that under the assumption of a positive generator the value process of a supersolution is a supermartingale. To view supersolutions as supermartingales is one of the key ideas in our approach.

Lemma 3.2. *Let g be a generator fulfilling* (POS)*, and $\xi \in L^0$ be a terminal condition such that $\xi^- \in L^1$. Let $(Y, Z) \in \mathcal{A}(\xi, g)$. Then $\xi \in L^1$, Y is a supermartingale, Z is unique and Y has the unique decomposition*

$$Y = Y_0 - A + M, \tag{3.6}$$

where M denotes the supermartingale $\int Z dW$ and A is a predictable, increasing, càdlàg process with $A_0 = 0$.

Proof. Relation (3.1), positivity of g, admissibility of Z and $\xi^- \in L^1$ imply $E[|Y_t|] < +\infty$, for all $t \in [0, T]$. Since $-\xi^- \leq \xi \leq Y_T$, we deduce that $\xi \in L^1$. Again, from (3.1), admissibility of Z and positivity of g we derive by taking conditional expectation, that $Y_s \geq E[Y_t \,|\, \mathcal{F}_s]$. Thus Y is a supermartingale with $Y_t \geq E\,[\xi \,|\, \mathcal{F}_t]$. Relation (3.1) implies further that there exist an increasing and càdlàg process K, with $K_0 = 0$, such that (3.6) holds with $A = \int g(Y, Z)ds + K$. Note that A is optional and therefore predictable due to the Brownian filtration, see [36, Corollary V.3.3]. Since Y is a càdlàg supermartingale the Doob-Meyer theorem, see [34, Theorem III.3.13], implies the unique decomposition $Y = Y_0 + \tilde{M} - \tilde{A}$, where $\tilde{M}$ is a local martingale and $\tilde{A}$ is an increasing process which is predictable, and $\tilde{M}_0 = \tilde{A}_0 = 0$. In our filtration every local martingale is continuous, see [34, Corollary IV.3.1, p. 187] and thus $\tilde{A}$ is càdlàg. Hence A and $\tilde{A}$ and M and $\tilde{M}$ are indistinguishable. Moreover, from the predictable representation property of local martingales and from $P(\bigcup_n \{\tau_n = T\}) = 1$, for $\tau^n = \inf\{t \geq 0 | |M_t| \geq n\} \wedge T$, we obtain the $P \otimes dt$-almost sure uniqueness of Z. □

The next proposition addresses the dependence of $\mathcal{A}\,(\xi, g)$ on ξ and g, and its impact on $\hat{\mathcal{E}}^g\,(\xi)$. The first two properties are crucial in the proof of the existence and uniqueness theorem in Section 4. The third item concerns the monotonicity of $\hat{\mathcal{E}}^g\,(\xi)$ with respect to ξ and g. Combined with the existence theorem, this yields in fact a comparison principle for minimal supersolutions of BSDEs. Finally, the last item concerns the cash (super/sub) additivity of the functional $\hat{\mathcal{E}}^g(\xi)$.

Proposition 3.3. *For $t \in [0, T]$, generators g, g' and terminal conditions $\xi, \xi' \in L^0$, it holds*

1. *the set $\{Y_t : (Y, Z) \in \mathcal{A}(\xi, g)\}$ is directed downwards.*

2. *assuming* (POS), $\xi^- \in L^1$ *and* $\mathcal{A}(\xi, g) \neq \emptyset$, *then for all* $\varepsilon > 0$, *there exists* $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ *such that* $\hat{\mathcal{E}}_t^g(\xi) \geq Y_t^\varepsilon - \varepsilon$.

3. (monotonicity) *if* $\xi' \leq \xi$ *and* $g'(y,z) \leq g(y,z)$, *for all* $y, z \in \mathbb{R} \times \mathbb{R}^{1\times d}$, *then* $\mathcal{A}(\xi', g') \supset \mathcal{A}(\xi, g)$ *and* $\hat{\mathcal{E}}_t^{g'}(\xi') \leq \hat{\mathcal{E}}_t^g(\xi)$.

4. (convexity) *if* $(y,z) \mapsto g(y,z)$ *is jointly convex, then* $\mathcal{A}(\lambda\xi + (1-\lambda)\xi', g) \supset \lambda\mathcal{A}(\xi, g) + (1-\lambda)\mathcal{A}(\xi', g)$, *for all* $\lambda \in (0,1)$, *and so*
$$\hat{\mathcal{E}}_t^g(\lambda\xi + (1-\lambda)\xi') \leq \lambda\hat{\mathcal{E}}_t^g(\xi) + (1-\lambda)\hat{\mathcal{E}}_t^g(\xi').$$

5. *for* $m \in L_t^0$,
   - (cash superadditivity) *assuming* (INC) *and* $m \geq 0$, *then* $\hat{\mathcal{E}}_t^g(\xi + m) \geq \hat{\mathcal{E}}_t^g(\xi) + m$.
   - (cash subadditivity) *assuming* (DEC), $m \geq 0$, *and the existence of* $(Y, Z) \in \mathcal{A}(\xi, g)$, *such that* $\mathcal{A}_t(Y_t + m, g) \neq \emptyset$, *then* $\hat{\mathcal{E}}_t^g(\xi + m) \leq \hat{\mathcal{E}}_t^g(\xi) + m$.
   - (cash additivity) *assuming that* g *does not depend on* y, *the existence of* $(Y, Z) \in \mathcal{A}(\xi, g)$, *such that* $\mathcal{A}_t(Y_t + m^+, g) \neq \emptyset$, *and the existence of* $(Y, Z) \in \mathcal{A}(\xi + m, g)$, *such that* $\mathcal{A}_t(Y_t + m^-, g) \neq \emptyset$, *then* $\hat{\mathcal{E}}_t^g(\xi + m) = \hat{\mathcal{E}}_t^g(\xi) + m$.

Proof. 1. Given $(Y^i, Z^i) \in \mathcal{A}(\xi, g)$, for $i = 1, 2$, we have to construct $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$, such that $\bar{Y}_t \leq \min\{Y_t^1, Y_t^2\}$. To this end, we define the stopping time
$$\tau = \inf\{s > t : Y_s^1 > Y_s^2\} \wedge T$$
and set $\bar{Y} = Y^1 1_{[0,\tau[} + Y^2 1_{[\tau,T[}$, $\bar{Y}_T = \xi$, and $\bar{Z} = Z^1 1_{[0,\tau]} + Z^2 1_{]\tau,T]}$. Since $Y_\tau^1 \geq Y_\tau^2$, Lemma 3.1 yields $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$ and by definition holds $\bar{Y}_t = \min\{Y_t^1, Y_t^2\}$.

2. In view of the first assertion, there exists a sequence $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ such that $(\tilde{Y}_t^n)$ decreases to $\hat{\mathcal{E}}_t^g(\xi)$. Set $Y^n = \tilde{Y}^1 1_{[0,t)} + \tilde{Y}^n 1_{[t,T]}$ and $Z^n = \tilde{Z}^1 1_{[0,t]} + \tilde{Z}^n 1_{(t,T]}$. From Lemma 3.1 follows that $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$ and (Y_t^n) decreases to $\hat{\mathcal{E}}_t^g(\xi)$ by construction. Lemma 3.2 implies that $\hat{\mathcal{E}}_t^g(\xi) \geq E[\xi^- | \mathcal{F}_t]$. Hence, given $\varepsilon > 0$, the sets $B^n = A^n \setminus A^{n-1} \in \mathcal{F}_t$, where $A^n = \{\hat{\mathcal{E}}_t^g(\xi) \geq Y_t^n - \varepsilon\}$ and $A^0 = \emptyset$, form a partition of Ω. Since (Y_t^n) is decreasing, it follows that $Y_t^1 1_{B^n} \geq Y_t^n 1_{B^n}$, for all $n \in \mathbb{N}$. Consequently, by means of Lemma 3.1, $(\bar{Y}, \bar{Z})$, defined as in (3.5), is an element of $\mathcal{A}(\xi, g)$ and $\hat{\mathcal{E}}_t^g(\xi) \geq \bar{Y}_t - \varepsilon$ by construction.

3. Follows from Definitions (3.3) and (3.4).

4. The joint convexity of g yields $(\lambda Y + (1-\lambda)Y', \lambda Z + (1-\lambda)Z') \in \mathcal{A}(\lambda\xi + (1-\lambda)\xi', g)$, for all $(Y, Z) \in \mathcal{A}(\xi, g)$, all $(Y', Z') \in \mathcal{A}(\xi', g)$, and all $\lambda \in (0,1)$. Hence, $\lambda\mathcal{A}(\xi, g) + (1-\lambda)\mathcal{A}(\xi', g) \subset \mathcal{A}(\lambda\xi + (1-\lambda)\xi', g)$ and in particular, $\hat{\mathcal{E}}_t^g(\lambda\xi + (1-\lambda)\xi') \leq \lambda\hat{\mathcal{E}}_t^g(\xi) + (1-\lambda)\hat{\mathcal{E}}_t^g(\xi')$.

5. Let us show the cash superadditivity. For $m \in L_t^0$ with $m \geq 0$, given $(Y, Z) \in \mathcal{A}(\xi + m, g)$, and $0 \leq s \leq t' \leq T$, it follows from (3.1) and (INC) that
$$Y_s - m1_{[t,T]}(s) - \int_s^{t'} g_u(Y_u - m1_{[t,T]}(u), Z_u)du + \int_s^{t'} Z_u dW_u$$
$$\geq Y_s - m1_{[t,T]}(s) - \int_s^{t'} g_u(Y_u, Z_u)du + \int_s^{t'} Z_u dW_u \geq Y_{t'} - m1_{[t,T]}(t').$$

Hence, $(Y - m1_{[t,T]}, Z) \in \mathcal{A}(\xi, g)$ and thus $\hat{\mathcal{E}}^g_t(\xi + m) - m \geq \hat{\mathcal{E}}^g_t(\xi)$. For the cash subadditivity the same argument yields

$$Y_s + m1_{[t,T]}(s) - \int_s^{t'} g_u(Y_u + m1_{[t,T]}(u), Z_u)du + \int_s^{t'} Z_u dW_u \geq Y_{t'} + m1_{[t,T]}(t'),$$

for all $t \leq s \leq t' \leq T$, and all $(Y, Z) \in \mathcal{A}(\xi, g)$. In order to apply our usual pasting argument we now need the assumption that $\mathcal{A}_t(Y_t + m, g) \neq \emptyset$. It provides $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}_t(Y_t + m, g)$ such that we may construct $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi + m, g)$, with $Y_t + m = \bar{Y}_t$ and thus $\hat{\mathcal{E}}^g_t(\xi) + m \geq \hat{\mathcal{E}}^g_t(\xi + m)$. The cash additivity in case where g is independent of y follows from $\hat{\mathcal{E}}^g_t(\xi) + m = \hat{\mathcal{E}}^g_t(\xi + m^+) - m^- = \hat{\mathcal{E}}^g_t(\xi + m + m^-) - m^- = \hat{\mathcal{E}}^g_t(\xi + m)$, since (DEC) and (INC) are simultaneously fulfilled. □

We now prove that for a positive generator $\hat{\mathcal{E}}^g(\xi)$ is in fact a supermartingale, which, in addition, dominates its right hand limit process. This is crucial for the proof of the existence and uniqueness theorem.

Proposition 3.4. *Let g be a generator fulfilling* (POS)*, and $\xi \in L^0$ be a terminal condition such that $\xi^- \in L^1$. Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$, then $\hat{\mathcal{E}}^g(\xi)$ is a supermartingale,*

$$\mathcal{E}^g_s(\xi) := \lim_{t \downarrow s, t \in \mathbb{Q}} \hat{\mathcal{E}}^g_t(\xi), \quad \textit{for all } s \in [0, T), \qquad \mathcal{E}^g_T(\xi) := \xi,$$

is a well-defined càdlàg supermartingale, and

$$\hat{\mathcal{E}}^g_s(\xi) \geq \mathcal{E}^g_s(\xi), \quad \textit{for all } s \in [0, T]. \tag{3.7}$$

Proof. Note first that $\hat{\mathcal{E}}^g(\xi)$ is adapted by definition. Furthermore, given $(Y, Z) \in \mathcal{A}(\xi, g) \neq \emptyset$, Lemma 3.2 implies $\xi \in L^1$ and $Y_t \geq E[\xi \mid \mathcal{F}_t]$. Hence $Y_t \geq \hat{\mathcal{E}}^g_t(\xi) \geq E[\xi \mid \mathcal{F}_t]$ and $\hat{\mathcal{E}}^g_t(\xi) \in L^1$. As for the supermartingale property and (3.7), fix $0 \leq s \leq t \leq T$. In view of Item 2 of Proposition 3.3, for all $\varepsilon > 0$, there exists $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ such that $\hat{\mathcal{E}}^g_s(\xi) \geq Y^\varepsilon_s - \varepsilon$. Due to (3.1) it follows

$$\begin{aligned}\hat{\mathcal{E}}^g_t(\xi) &\leq Y^\varepsilon_t \leq Y^\varepsilon_s - \int_s^t g_u(Y^\varepsilon_u, Z^\varepsilon_u)du + \int_s^t Z^\varepsilon_u dW_u \\ &\leq \hat{\mathcal{E}}^g_s(\xi) - \int_s^t g_u(Y^\varepsilon_u, Z^\varepsilon_u)du + \int_s^t Z^\varepsilon_u dW_u + \varepsilon \leq \hat{\mathcal{E}}^g_s(\xi) + \int_s^t Z^\varepsilon_u dW_u + \varepsilon. \end{aligned} \tag{3.8}$$

Taking conditional expectation on both sides with respect to $\mathcal{F}_s$ and the supermartingale property of $\int Z^\varepsilon dW$ yields $\hat{\mathcal{E}}^g_s(\xi) \geq E[\hat{\mathcal{E}}^g_t(\xi) \mid \mathcal{F}_s]$, and so $\hat{\mathcal{E}}^g(\xi)$ is a supermartingale. That $\mathcal{E}^g(\xi)$ is well-defined càdlàg supermartingale follows from Karatzas and Shreve [26, Proposition 1.3.14]. Finally, (3.7) follows directly from (3.8) and the definition of $\mathcal{E}^g(\xi)$. □

Remark 3.5. The previous proposition suggests to consider the càdlàg supermartingale $\mathcal{E}^g(\xi)$ as a candidate for the value process of the minimal supersolution. Note further that, if $\mathcal{E}^g(\xi)$ is the value process of the minimal supersolution it is a modification of $\hat{\mathcal{E}}^g(\xi)$ by definition. ♦

The final result of this Section shows that our setup allows to derive various properties that are important in the context of non-linear expectations and dynamic risk measures. In particular, we prove that $\mathcal{E}^g(\xi)$, if it is the value process of the minimal supersolution, fulfills the flow-property and, under the additional assumption $g(y,0) = 0$, for all $y \in \mathbb{R}$, we show projectivity, with time-consistency as a special case. In the context of BSDE solutions such properties were first established in [29], for the case of Lipschitz generators. For dynamic risk measures the (strong) time-consistency has been investigated in discrete time in [8, 19] as well as in continuous time in [4, 9], for instance.

Proposition 3.6. *For $t \in [0,T]$, generator g and terminal condition $\xi \in L^0$, it holds*

1. $\hat{\mathcal{E}}^g_{s,T}(\xi) \leq \hat{\mathcal{E}}^g_{s,t}(\hat{\mathcal{E}}^g_{t,T}(\xi))$, *for all $0 \leq s \leq t$. Suppose that $\mathcal{E}^g(\xi)$ is a minimal supersolution, then the* flow-property *holds, that is*
$$\mathcal{E}^g_{s,T}(\xi) = \mathcal{E}^g_{s,t}(\mathcal{E}^g_{t,T}(\xi)), \quad \text{for all } 0 \leq s \leq t. \tag{3.9}$$
2. *if $g(y,0) = 0$, for all $y \in \mathbb{R}$, then $\hat{\mathcal{E}}^g_s\left(\hat{\mathcal{E}}^g_t(\xi)\right) \leq \hat{\mathcal{E}}^g_s(\xi)$, for all $0 \leq s \leq t$. Assuming* (POS), *$\xi^- \in L^1$, and supposing that $\mathcal{E}^g(\xi)$ is a minimal supersolution, then $\mathcal{E}^g(\xi)$ is* time-consistent, *that is*
$$\mathcal{E}^g_s(\mathcal{E}^g_t(\xi)) = \mathcal{E}^g_s(\xi), \quad \text{for all } 0 \leq s \leq t. \tag{3.10}$$
3. *assuming* (POS), *$g(y,0) = 0$, for all $y \in \mathbb{R}$, $\xi^- \in L^1$, and $\mathcal{E}^g(\xi)$ is a minimal supersolution, then the* projectivity *holds, that is*
$$\mathcal{E}^g_s(1_A\mathcal{E}^g_t(\xi)) = \mathcal{E}^g_s(1_A\xi), \quad \text{for all } 0 \leq s \leq t \text{ and } A \in \mathcal{F}_t. \tag{3.11}$$

Proof. 1. Fix $0 \leq s \leq t$. Obviously, $(Y_s, Z_s)_{s\in[0,t]} \in \mathcal{A}_t(\hat{\mathcal{E}}^g_{t,T}(\xi), g)$, for all $(Y,Z) \in \mathcal{A}_T(\xi, g)$. Hence $\hat{\mathcal{E}}^g_{s,t}(\hat{\mathcal{E}}^g_{t,T}(\xi)) \leq \hat{\mathcal{E}}^g_{s,T}(\xi)$. Suppose now that $\mathcal{E}^g_{\cdot,T}(\xi)$ is a minimal supersolution with corresponding admissible control process $\hat{Z} \in \mathcal{L}$. For all $(Y,Z) \in \mathcal{A}_t(\mathcal{E}^g_{t,T}(\xi), g)$, holds $Y_t \geq \mathcal{E}^g_{t,T}(\xi)$ and, with the same argumentation as in Lemma 3.1, we can paste in a monotone way to show that $(\bar{Y}, \bar{Z}) \in \mathcal{A}_T(\xi, g)$, where $\bar{Y} = Y1_{[0,t[} + \mathcal{E}^g_{\cdot,T}(\xi)1_{[t,T]}$ and $\bar{Z} = Z1_{[0,t]} + \hat{Z}1_{]t,T]}$. Thus, by definition, $\mathcal{E}^g_{s,t}(\mathcal{E}^g_{t,T}(\xi)) \geq \mathcal{E}^g_{s,T}(\xi)$.

2. Given $(Y,Z) \in \mathcal{A}(\xi, g)$, we define $\bar{Y} = Y1_{[0,t[} + \hat{\mathcal{E}}^g_t(\xi)1_{[t,T]}$ and $\bar{Z} = Z1_{[0,t]}$. Since $Y_t \geq \hat{\mathcal{E}}^g_t(\xi)$ and $g(y,0) = 0$, it is straightforward to verify that $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\hat{\mathcal{E}}^g_t(\xi), g)$. From $Y_s \geq \bar{Y}_s$, for all $s \in [0,t]$, follows that $\hat{\mathcal{E}}^g_s(\hat{\mathcal{E}}^g_t(\xi)) \leq \hat{\mathcal{E}}^g_s(\xi)$, for all $s \in [0,t]$. The case where $\mathcal{E}^g(\xi)$ is a minimal supersolution and Assumption (POS) holds, follows from (3.11) for $A = \Omega$.

3. Fix $A \in \mathcal{F}_t$. Suppose that $\mathcal{E}^g(\xi)$ is a minimal supersolution with corresponding control process $\hat{Z}$. Then, from $\xi^- \in L^1$ and Lemma 3.2 follows that $\mathcal{E}^g(\xi)$ is a supermartingale and $\xi \in L^1$.

Given $(Y,Z) \in \mathcal{A}(1_A\mathcal{E}^g_t(\xi), g)$, it follows from $(1_A\mathcal{E}^g_t(\xi))^- \in L^1$ and Lemma 3.2 that $Y_t \geq E[1_A\mathcal{E}^g_t(\xi) \mid \mathcal{F}_t] = 1_A\mathcal{E}^g_t(\xi)$. Since $g(y,0) = 0$, it is straightforward to check that $\tilde{Y} = Y1_{[0,t[} + \mathcal{E}^g_t(\xi)1_A1_{[t,T]}$, and $\tilde{Z} = Z1_{[0,t]}$ is such that $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}(1_A\mathcal{E}^g_t(\xi), g)$. We can henceforth assume that $Y_s = 1_A\mathcal{E}^g_t(\xi)$, for all $s \geq t$. Now, we define $\bar{Y} = Y1_{[0,t[} + \mathcal{E}^g(\xi)1_A1_{[t,T]}$ and $\bar{Z} = Z1_{[0,t]} + \hat{Z}1_A1_{]t,T]}$. For $0 \leq s < t \leq t' \leq T$ holds

$$\begin{aligned}\bar{Y}_s - \int_s^{t'} g\left(\bar{Y}_u, \bar{Z}_u\right)du + \int_s^{t'} \bar{Z}_u dW_u &\geq Y_t + \left(-\int_t^{t'} g_u\left(\mathcal{E}^g_u(\xi), \hat{Z}_u\right)du + \int_t^{t'} \hat{Z}_u dW_u\right)1_A \\ &\geq \left(\mathcal{E}^g_t(\xi) - \int_t^{t'} g_u\left(\mathcal{E}^g_u(\xi), \hat{Z}_u\right)du + \int_t^{t'} \hat{Z}_u dW_u\right)1_A \geq \mathcal{E}^g_{t'}(\xi)1_A.\end{aligned}$$

Hence, for all $0 \leq s \leq t' \leq T$, holds

$$\bar{Y}_s - \int\limits_s^{t'} g\left(\bar{Y}_u, \bar{Z}_u\right) du + \int\limits_s^{t'} \bar{Z}_u dW_u \geq Y_{t'} 1_{\{t'<t\}} + \mathcal{E}^g_{t'}(\xi) 1_A 1_{\{t \leq t'\}} = \bar{Y}_{t'}$$

and $\bar{Y}_T = 1_A \xi$, which implies that $(\bar{Y}, \bar{Z}) \in \mathcal{A}(1_A \xi, g)$. Since $\bar{Y}_s = Y_s$, for all $s \leq t$, we deduce $\mathcal{E}^g_s(1_A \xi) \leq \mathcal{E}^g_s(1_A \mathcal{E}_t(\xi))$.

On the other hand, consider $(Y, Z) \in \mathcal{A}(1_A \xi, g)$. From $Y_t \geq E[1_A \xi \,|\, \mathcal{F}_t] = 1_A E[\xi \,|\, \mathcal{F}_t]$, we obtain $Y_t 1_{A^c} \geq 0$. Since $\mathcal{E}^g(\xi)$ is a minimal supersolution, it follows that $Y_t \geq \mathcal{E}^g_t(\xi) 1_A$. Indeed, let $B = \{Y_t < \mathcal{E}^g_t(\xi) 1_A\}$, then $Y_t 1_{A^c} \geq 0$ implies $B \subset A$. Consequently, by similar arguments as in Lemma 3.1, the processes $\tilde{Y} = \mathcal{E}^g(\xi)(1_{[0,t[} + 1_{B^c} 1_{[t,T]}) + Y 1_B 1_{[t,T]}$ and $\tilde{Z} = \hat{Z}(1_{[0,t[} + 1_{B^c} 1_{[t,T]}) + Z 1_B 1_{[t,T]}$ are such that $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}(\xi, g)$, which implies $P[B] = 0$. It is also straightforward to check that $\tilde{Y} = Y 1_{[0,t[} + \mathcal{E}^g(\xi) 1_A 1_{[t,T]}$ and $\tilde{Z} = Z 1_{[0,t]} + \hat{Z} 1_{(t,T]} 1_A$ are such that $(\tilde{Y}, \tilde{Z}) \in \mathcal{A}(1_A \xi, g)$. Thus we can assume that $Y_t = 1_A \mathcal{E}^g_t(\xi)$. Defining $\bar{Y} = Y 1_{[0,t[} + \mathcal{E}^g_t(\xi) 1_A 1_{[t,T]}$ and $\bar{Z} = Z 1_{[0,t]}$, it holds $(\bar{Y}, \bar{Z}) \in \mathcal{A}(1_A \mathcal{E}^g_t(\xi), g)$. Thus $\mathcal{E}^g_s(1_A \mathcal{E}^g_t(\xi)) \leq \mathcal{E}^g_s(1_A \xi)$, since $\bar{Y}_s = Y_s$, for all $s \leq t$. $\square$

4 Existence, Uniqueness and Stability

In this section, we give conditions, which guarantee the existence and uniqueness of a minimal supersolution. We show that the corresponding value process is given by $\mathcal{E}^g(\xi)$. Moreover, we analyze the stability of $\hat{\mathcal{E}}^g(\xi)$ with respect to perturbations of the terminal condition or the generator. In addition to the assumptions (POS) and (INC) or (DEC) introduced above, we require convexity of g in the control variable and joint lower semicontinuity. To that end, we say that a generator g is

(CON) convex, if $g(y, \lambda z + (1-\lambda) z') \leq \lambda g(y, z) + (1-\lambda) g(y, z')$, for all $y \in \mathbb{R}$, all $z, z' \in \mathbb{R}^{1\times d}$ and all $\lambda \in (0, 1)$.

(LSC) if $(y, z) \mapsto g(y, z)$ is lower semicontinuous.

4.1 Existence and Uniqueness of Minimal Supersolutions

The following theorem on existence and uniqueness of a minimal supersolution is the first main result of this paper.

Theorem 4.1. *Let g be a generator fulfilling* (POS), (LSC), (CON) *and either* (INC) *or* (DEC) *and $\xi \in L^0$ be a terminal condition, such that $\xi^- \in L^1$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique minimal supersolution $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g)$. Moreover, $\mathcal{E}^g(\xi)$ is the value process of the minimal supersolution, that is $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Note that, under the assumptions of Theorem 4.1, Remark 3.5 implies that the process $\mathcal{E}^g(\xi)$ is a modification of $\hat{\mathcal{E}}^g(\xi)$. Further, in the context of finding minimal elements in some set the assumption $\mathcal{A}(\xi, g) \neq \emptyset$ is quite standard, see [30] for an example in the setting of minimal supersolutions. However, let us point out that in many applications $\mathcal{A}(\xi, g) \neq \emptyset$ might be guaranteed by specific model assumptions, see for instance an example on utility maximization in Heyne [24]. It might also be automatically granted under further assumptions, see Cheridito and Stadje [7], or for instance if the BSDE $Y_t - \int_t^T g_s(Y_s, Z_s) ds + \int_t^T Z_s dW_s = \hat{\xi}$ has a solution $(Y, Z) \in \mathcal{S} \times \mathcal{L}$, such that Z is admissible. In the latter case, $\mathcal{A}(\xi, g) \neq \emptyset$, for all $\xi \in L^0$ such that $\xi^- \in L^1$, with $\hat{\xi} \geq \xi$.

Proof. *Step 1: Uniqueness.* Given $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, the definition of $\mathcal{E}^g(\xi)$ implies that for any other supersolution $(Y, Z') \in \mathcal{A}(\xi, g)$ holds $\mathcal{E}^g_t(\xi) \leq Y_t$, for all $t \in [0, T]$. The uniqueness of $\hat{Z}$ follows as in Lemma 3.2.

The remainder of the proof provides existence of $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.

Step 2: Construction of an approximating sequence. For any $n, i \in \mathbb{N}$, let $t_i^n = iT/2^n$. There exist $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$ such that

$$\hat{\mathcal{E}}^g_{t_i^n}(\xi) \geq Y^n_{t_i^n} - 1/n, \quad \text{for all } n \in \mathbb{N}, \text{ and all } i = 0, \ldots, 2^n - 1, \tag{4.1}$$

and

$$Y^n_t \geq Y^{n+1}_t, \quad \text{for all } t \in [0, T], \text{ and all } n \in \mathbb{N}. \tag{4.2}$$

Indeed, by means of Proposition 3.3.2, for each $n \in \mathbb{N}$, we may select a family $((Y^{n,i}, Z^{n,i}))_{i=0,\ldots,2^n-1}$ in $\mathcal{A}(\xi, g)$, such that $\hat{\mathcal{E}}^g_{t_i^n}(\xi) \geq Y^{n,i}_{t_i^n} - 1/n$, $i = 0, \ldots, 2^n - 1$. We suitably paste this family in order to obtain (4.1). We start with

$$\bar{Y}^{n,0} = Y^{n,0}, \quad \bar{Z}^{n,0} = Z^{n,0},$$

and continue by recursively setting, for $i = 1, \ldots, 2^n - 1$,

$$\begin{aligned} \bar{Y}^{n,i} &= \bar{Y}^{n,i-1} 1_{[0,\tau_i^n[} + Y^{n,i} 1_{[\tau_i^n, T[}, \quad \bar{Y}^{n,i}_T = \xi, \\ \bar{Z}^{n,i} &= \bar{Z}^{n,i-1} 1_{[0,\tau_i^n]} + Z^{n,i} 1_{]\tau_i^n, T]}, \end{aligned}$$

where τ_i^n are stopping times given by $\tau_i^n = \inf\{t > t_i^n : \bar{Y}^{n,i-1}_t > Y^{n,i}_t\} \wedge T$. From the definition of these stopping times and Lemma 3.1 follows that the pairs $(\bar{Y}^{n,i}, \bar{Z}^{n,i})$, $i = 0, \ldots, 2^n - 1$, are elements of $\mathcal{A}(\xi, g)$. Hence, the sequence

$$((Y^n, Z^n) := (\bar{Y}^{n,2^n-1}, \bar{Z}^{n,2^n-1}))$$

fulfills (4.1) by construction. Note that $((Y^n, Z^n))$ is not necessarily monotone in the sense of (4.2). However, this can be achieved by pasting similarly. More precisely, we choose

$$\bar{Y}^1 = Y^1, \quad \bar{Z}^1 = Z^1,$$

and continue by recursively setting, for $n \in \mathbb{N}$,

$$\begin{aligned} \bar{Y}^n &= \sum_{i=0}^{2^n-1} \left(Y^n 1_{[t_i^n, \tau_i^n[} + \bar{Y}^{n-1} 1_{[\tau_i^n, t_{i+1}^n[} \right), \quad \bar{Y}^n_T = \xi, \\ \bar{Z}^n &= \sum_{i=0}^{2^n-1} \left(Z^n 1_{]t_i^n, \tau_i^n]} + \bar{Z}^{n-1} 1_{]\tau_i^n, t_{i+1}^n]} \right), \end{aligned}$$

where τ_i^n are stopping times given by $\tau_i^n = \inf\{t > t_i^n : Y^n_t > \bar{Y}^{n-1}_t\} \wedge t_{i+1}^n$, for $i = 0, \ldots, 2^n - 1$. By construction $((\bar{Y}^n, \bar{Z}^n))$ fulfills both (4.1) and (4.2), and $((\bar{Y}^n, \bar{Z}^n)) \subset \mathcal{A}(\xi, g)$ with Lemma 3.1.

Step 3: Bound on $\int Z^n dW$. We now take the sequence $((Y^n, Z^n))$ fulfilling (4.1) and (4.2) and provide an inequality which will enable us to use compactness arguments for (Z^n) later in the proof. More precisely, we argue that, for all $n \in \mathbb{N}$, holds

$$\left| \int_0^t Z^n_s dW_s \right| \leq B^n_t := \left| Y^1_t \right| + E\left[\xi^- \mid \mathcal{F}_t \right] + E\left[\xi^- \right] + \left| Y^1_0 \right| + A^n_t, \tag{4.3}$$

for all $t \in [0,T]$, where A_t^n is the positive increasing process defined in Lemma 3.2. Moreover, it holds

$$E[A_T^n] \le Y_0^1 - E[\xi].$$

Indeed, by the same arguments as in the proof of Lemma 3.3.2, recall $Y_0^n \le Y_0^1$, it follows

$$\int_0^t Z_s^n dW_s \ge -E\left[\xi^- \mid \mathcal{F}_t\right] - Y_0^1. \tag{4.4}$$

On the other hand, from $Y_t^n \le Y_t^1$ and $-Y_0^n \le E[\xi^-]$, recall Lemma 3.2, it follows

$$\int_0^t Z_s^n dW_s \le Y_t^1 + A_t^n - Y_0^n \le Y_t^1 + A_t^n + E[\xi^-]. \tag{4.5}$$

Combining (4.4) and (4.5) yields (4.3). The L^1 bound on A^n follows from $Y_0^n - A_T^n + \int_0^T Z_s^n dW_s = \xi$, $Y_0^1 \ge Y_0^n$, and the supermartingal property of $\int Z^n dW$.

Note that if $(B_T^{n,*})$ in (4.3) were bounded in L^1, then, by means of the Burkholder-Davis-Gundy inequality, (Z^n) would be a bounded sequence in $\mathcal{L}^1$ and we could apply [11, Theorem A] to find a sequence in the asymptotic convex hull of (Z^n) converging in $\mathcal{L}^1$ and $P \otimes dt$-almost surely along some localizing sequence of stopping times to some limit $Z \in \mathcal{L}^1$. Here, even if $(A_T^{n,*}) = (A_T^n)$ is uniformly bounded, this is however not necessarily the case for $Y_T^{1,*}$ and $(E[\xi^- \mid \mathcal{F}_\cdot])_T^*$, and this is the reason why we introduce the following localization.

Step 4: First localization. Due to our Brownian setting and since $\xi^- \in L^1$, we know that the martingale $E[\xi^- \mid \mathcal{F}_\cdot]$, has a continuous version, see [36, Theorem V.3.5]. Moreover, Y^1 is a càdlàg supermartingale and thus we may take the localising sequence

$$\sigma_k = \inf\left\{t > 0 : \left|Y_t^1\right| + E\left[\xi^- \mid \mathcal{F}_t\right] > k\right\} \wedge T, \quad k \in \mathbb{N}, \tag{4.6}$$

which is independent of $n \in \mathbb{N}$. For a fixed $k \in \mathbb{N}$, Inequality (4.3) yields

$$\left(\int Z^n 1_{[0,\sigma_k]} dW\right)_T^* \le B^{k,n}, \quad \text{for all } n \in \mathbb{N}, \tag{4.7}$$

where $B^{k,n} = \left|Y_0^1\right| + E[\xi^-] + k + A_T^n$. Due to $E[A_T^n] \le Y_0^1 - E[\xi]$ we have

$$\sup_{n \in \mathbb{N}} E\left[B^{k,n}\right] < \infty. \tag{4.8}$$

Since $(B^{k,n})_{n \in \mathbb{N}}$ is a sequence of positive random variables we may apply [10, Lemma A1.1]. It provides a sequence $(\tilde{B}^{k,n})_{n \in \mathbb{N}}$ in the asymptotic convex hull of $(B^{k,n})_{n \in \mathbb{N}}$, which converges almost surely to a random variable $\tilde{B}^k \ge 0$. The $\tilde{B}^{k,n}$ inherit the integrability of the $B^{k,n}$ and we can conclude with Fatou's lemma that

$$E[\tilde{B}^k] < \infty. \tag{4.9}$$

Let $\tilde{Z}^{k,n}$ be the convex combination of (Z^n) corresponding to $\tilde{B}^{k,n}$ so that

$$\left(\int \tilde{Z}^{k,n} 1_{[0,\sigma_k]} dW\right)_T^* \le \tilde{B}^{k,n}, \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Step 5: Second localization. The next two steps follow some known compactness arguments, which, in the case of $\mathcal{L}^1$, can be found in [11]. For the sake of completeness we develop the argumentation. Given an $m \in \mathbb{N}$, we start by taking a fast subsequence $(\tilde{B}^{k,m,n})_{n\in\mathbb{N}}$ of $(\tilde{B}^{k,n})_{n\in\mathbb{N}}$ converging in probability to $\tilde{B}^k$. More precisely, we choose $(\tilde{B}^{k,m,n})_{n\in\mathbb{N}}$ such that

$$P\left[\left|\tilde{B}^{k,m,n} - \tilde{B}^k\right| \geq 1\right] \leq \frac{2^{-n}}{m}. \tag{4.11}$$

Consider now the stopping time $\tau^{k,m}$ given by

$$\tau^{k,m} = \inf\left\{t \geq 0 : \left(\int \tilde{Z}^{k,m,n} 1_{[0,\sigma_k]} dW\right)^*_t \geq m, \text{ for some } n \in \mathbb{N}\right\} \wedge T, \tag{4.12}$$

where the sequence $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]})_{n\in\mathbb{N}}$ is the subsequence of $(\tilde{Z}^{k,n}1_{[0,\sigma_k]})_{n\in\mathbb{N}}$ corresponding to the fast subsequence $(\tilde{B}^{k,m,n})_{n\in\mathbb{N}}$. The definition of $\tau^{k,m}$ as well as the Burkholder-Davis-Gundy inequality imply that the sequence of processes $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]})_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^2$. The Alaoglu-Bourbaki theorem and the Eberlein-Šmulian theorem in the Banach space $\mathcal{L}^2$ imply the existence of $\hat{Z}^{k,m} \in \mathcal{L}^2$, such that, up to a subsequence, $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]})_{n\in\mathbb{N}}$ converges weakly to $\hat{Z}^{k,m}$. As a consequence of the Hahn-Banach theorem, there exists a sequence in the asymptotic convex hull of $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]})_{n\in\mathbb{N}}$, again denoted with $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]})_{n\in\mathbb{N}}$, which converges in $\mathcal{L}^2$ to $\hat{Z}^{k,m}$. By taking another subsequence we also have the $P \otimes dt$-almost sure convergence.

Step 6: $(\tau^{k,m})_{m\in\mathbb{N}}$ is a localizing sequence of stopping times. We estimate as follows

$$\begin{aligned}
P\left[\tau^{k,m} = T\right] &= P\left[\left(\int \tilde{Z}^{k,m,n}1_{[0,\sigma_k]}dW\right)^*_T < m \text{ , for all } n \in \mathbb{N}\right] \\
&\geq 1 - P\left[\tilde{B}^{k,m,n} \geq m \text{ , for some } n \in \mathbb{N}\right] \\
&\geq 1 - P\left[\left\{\left|\tilde{B}^{k,m,n} - \tilde{B}^k\right| \geq 1 \text{ , for some } n \in \mathbb{N}\right\} \cup \left\{\tilde{B}^k > m-1\right\}\right] \\
&\geq 1 - \sum_n P\left[\left|\tilde{B}^{k,m,n} - \tilde{B}^k\right| \geq 1\right] - P\left[\tilde{B}^k > m-1\right] \\
&\geq 1 - \frac{1}{m} - \frac{E\left[\tilde{B}^k + 1\right]}{m} \xrightarrow[m\to\infty]{} 1,
\end{aligned}$$

where we used (4.10) in the second line and (4.11), the Markov inequality and the fact that $E[\tilde{B}^k] < \infty$ in the last one.

Step 7: Construction of the candidate $\hat{Z}$. For given $k, m > 0$, we constructed in Step 5 the process $\hat{Z}^{k,m}$ as the $\mathcal{L}^2$ and $P \otimes dt$-almost sure limit of a sequence in the asymptotic convex hull of $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]})_{n\in\mathbb{N}}$. With $(\tilde{B}^{k,m,n})_{n\in\mathbb{N}}$ we denote the corresponding subsequence of convex combinations of $(\tilde{B}^{k,m,n})_{n\in\mathbb{N}}$ and note that $(\int \tilde{Z}^{k,m,n}1_{[0,\sigma_k]}dW)^*_T \leq \tilde{B}^{k,m,n}$, for all $n \in \mathbb{N}$, as in (4.10). Hence, by the same procedure as in Step 5, we can find, for $m' > m$, a fast subsequence $(\tilde{Z}^{k,m',n}1_{[0,\sigma_k]})_{n\in\mathbb{N}}$ in the asymptotic convex hull of $(\tilde{Z}^{k,m,n}1_{[0,\sigma_k]})_{n\in\mathbb{N}}$ and a $\hat{Z}^{k,m'} \in \mathcal{L}^2$ such that $(\tilde{Z}^{k,m',n}1_{[0,\sigma_k]}1_{[0,\tau^{k,m'}]})_{n\in\mathbb{N}}$ converges in $\mathcal{L}^2$ and $P \otimes dt$-almost surely to $\hat{Z}^{k,m'}$. We iterate this procedure and define $(\tilde{Z}^{k,n})_{n\in\mathbb{N}}$ as the diagonal sequence $\tilde{Z}^{k,n} = \tilde{Z}^{k,n,n}$ and $\hat{Z}^k$ as

$$Z^k_0 = 0, \quad \hat{Z}^k = \sum_{m=1}^{\infty} \hat{Z}^{k,m} 1_{]\tau^{k,m-1},\tau^{k,m}]}. \tag{4.13}$$

From $\hat{Z}^{k,m'}1_{[0,\tau^{k,m}]} = \hat{Z}^{k,m}$, for $m' > m$, follows that $(\tilde{Z}^{k,n}1_{[0,\sigma_k]}1_{[0,\tau^{k,n}]})_{n\in\mathbb{N}}$ converges in $\mathcal{L}^2$ and $P\otimes dt$-almost surely to $\hat{Z}^k$. With the sequence $(\tilde{Z}^{k,n})_{n\in\mathbb{N}}$ and the process $\hat{Z}^k$ at hand, we now diagonalize our program above with respect to k and n. As before, we get a diagonal sequence $\tilde{Z}^n = \tilde{Z}^{n,n}$, and a process $\hat{Z}$ given by

$$\hat{Z}_0 = 0, \quad \hat{Z} = \sum_{k=1}^{\infty} 1_{]\sigma_{k-1},\sigma_k]}\hat{Z}^k, \tag{4.14}$$

such that

$$\tilde{Z}^n 1_{[0,\tau_n]} \xrightarrow[n\to\infty]{P\otimes dt\text{-almost surely}} \hat{Z}, \tag{4.15}$$

for $\tau_n = \sigma_n \wedge \tau^{n,n}$, where σ_n and $\tau^{n,n}$ are as in (4.6) and (4.12), respectively. For later reference, note that by construction holds $\hat{Z}^{k',m}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]} = \hat{Z}^{k,m}$, as soon as $k' \geq k$ and also $\hat{Z}1_{[0,\sigma_k]}1_{[0,\tau^{k,m}]} = \hat{Z}^{k,m}$. Likewise $(\tilde{Z}^n 1_{[0,\tau_l]})_{n\in\mathbb{N}}$ converges in $\mathcal{L}^2$ and $P \otimes dt$-almost surely to $\hat{Z}^{l,l}$. This yields, via the Burkholder-Davis-Gundy inequality, up to a subsequence,

$$\int_0^{t\wedge\tau_l} \tilde{Z}^n_s dW_s \xrightarrow[n\to+\infty]{} \int_0^{t\wedge\tau_l} \hat{Z}_s dW_s, \quad \text{for all } t \in [0,T], P\text{-almost surely.} \tag{4.16}$$

Hence, diagonalizing yields

$$\int_0^{t} \tilde{Z}^n_s dW_s \xrightarrow[n\to+\infty]{} \int_0^{t} \hat{Z}_s dW_s, \quad \text{for all } t \in [0,T], P\text{-almost surely.} \tag{4.17}$$

Step 8: Monotone convergence to $\mathcal{E}^g(\xi)$. Let $\tilde{Y}_t = \lim_n Y^n_t$, for $t \in [0,T]$, be the pointwise monotone limit of the sequence (Y^n). By monotone convergence $\tilde{Y}$ is a supermartingale and, since our filtration is right-continuous, by standard arguments we may define the càdlàg supermartingale $\hat{Y}$ by setting $\hat{Y}_t = \lim_{s\downarrow t, s\in\mathbb{Q}} \tilde{Y}_s$, for all $t \in [0,T)$, and $\hat{Y}_T = \xi$. By construction $\tilde{Y}_{t^i_n} = \hat{\mathcal{E}}^g_{t^i_n}(\xi)$. Hence, $\hat{Y}_t = \mathcal{E}^g_t(\xi)$, for all $t \in [0,T]$, and

$$Y^n_t \geq \tilde{Y}_t \geq \hat{\mathcal{E}}^g_t(\xi) \geq \mathcal{E}^g_t(\xi) \geq E\left[\xi \mid \mathcal{F}_t\right], \tag{4.18}$$

where the third inequality follows from Proposition 3.4. Now, the process $\mathcal{E}^g(\xi)$ is the natural candidate for the value process of the minimal supersolution for two reasons. It is càdlàg and it is dominated by $\hat{\mathcal{E}}^g(\xi)$ as (4.18) shows. However, it is not clear a priori that the sequence (Y^n) converges to $\mathcal{E}^g(\xi)$ in some suitable sense. Taking into account the additional structure provided by the supermartingale property of the Y^n we can prove nonetheless

$$\mathcal{E}^g(\xi) = \hat{Y} = \lim_{n\to\infty} Y^n, \quad P \otimes dt\text{-almost surely.} \tag{4.19}$$

To see this note first that by right continuity the limit $\tilde{Y}_t = \lim_n Y^n_t$ is defined, for all $t \in [0,T]$, P-almost surely. We now consider the sequence $((\tilde{Y}^n, \tilde{Z}^n))$ in the asymptotic convex hull of (Y^n, Z^n), which corresponds to the sequence $(\tilde{Z}^n)$ constructed in Step 7. From the decomposition of the Y^n, see Lemma 3.2, we obtain that $\tilde{Y}^n_t = \tilde{Y}^n_0 - \tilde{A}^n_t + \tilde{M}^n_t$, for all $t \in [0,T]$. Since, $(\tilde{Y}^n_t)$ and $(\tilde{M}^n_t)$ converge for all $t \in [0,T]$, P-almost surely, the sequence $(\tilde{A}^n_t)$ also converges, that is there exists an increasing positive integrable process $\tilde{A}$, such that $\lim_{n\to\infty} \tilde{A}^n_t = \tilde{A}_t$, for all $t \in [0,T]$, P-almost surely. Thus $\tilde{Y}_t = \tilde{Y}_0 - \tilde{A}_t + \tilde{M}_t$, for all $t \in [0,T]$. Consequently, the jumps of $\tilde{Y}$ are given by the countably many jumps of the increasing process $\tilde{A}$, which implies

$$\hat{Y}_t = \tilde{Y}_0 - \lim_{s\downarrow t, s\in\mathbb{Q}} \tilde{A}_s + \tilde{M}_t, \qquad \text{for all } t \in [0,T), \qquad \hat{Y}_T = \xi.$$

Moreover, the jump times of the càdlàg process $\hat{Y}$ are exhausted by a sequence of stopping times $(\rho^j) \subset \mathcal{T}$, which coincide with the jump times of $\tilde{A}$. Therefore, $\hat{Y} = \tilde{Y}$, $P \otimes dt$-almost surely, which implies (4.19).

Step 9: Verification. Let us now show that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, which, by means of (4.18), would end the proof. We start with the verification of (3.1) under the Assumption (INC). Due to (4.19) there exists a set $B \subset \Omega \times [0,T]$ with $P \otimes dt(B^c) = 0$, such that $\mathcal{E}^g_t(\xi)(\omega) = \lim_{n\to\infty} Y^n_t(\omega)$, for all $(\omega, t) \in B$. Hence, there exists a set $A \subset \{\omega : (\omega, t) \in B, \text{ for some } t\}$, with $P(A) = 1$, such that, for all $\omega \in A$ the set $I(\omega) = \{t \in [0,T] : (\omega, t) \in B\}$ is a Lebesque set of measure T and $\mathcal{E}^g_t(\xi)(\omega) = \lim_{n\to\infty} Y^n_t(\omega)$, for all $t \in I(\omega)$. In the following we suppress the dependence of I on ω and just keep in mind that s and t may depend on ω. Let $s, t \in I$ with $s \leq t$. By using (4.17), the $P \otimes dt$-almost sure convergence of $\tilde{Z}^n 1_{[0,\tau^n]}$ to $\hat{Z}$, and Fatou's lemma we obtain

$$\begin{aligned}
&\mathcal{E}^g_s - \int_s^t g_u(\mathcal{E}^g_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \\
&\qquad\geq \limsup_n \left(\tilde{Y}^n_s - \int_s^t g_u(\mathcal{E}^g_u, \tilde{Z}^n_u 1_{[0,\tau^n]}(u))du + \int_s^t \tilde{Z}^n_u dW_u \right), \quad (4.20)
\end{aligned}$$

where $\tilde{Y}^n$ denotes the convex combination of (Y^n) corresponding to $\tilde{Z}^n$. We denote by $\lambda^{(n)}_i$, $n \leq i \leq M^{(n)}$, $\lambda^{(n)}_i \geq 0, \sum_i \lambda^{(n)}_i = 1$ the convex weights of $\tilde{Z}^n$. Since our generator fullills (CON), and since, for n large enough, we have $\tilde{Z}^n_u 1_{[0,\tau^n]}(u) = \tilde{Z}^n_u$, for all $s \leq u \leq t$, we may further estimate the above by

$$\mathcal{E}^g_s - \int_s^t g_u(\mathcal{E}^g_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda^{(n)}_i \left(Y^i_s - \int_s^t g_u(\mathcal{E}^g_u, Z^i_u)du + \int_s^t Z^i_u dW_u \right).$$

Since $Y^i_t \geq \hat{\mathcal{E}}^g_t(\xi) \geq \mathcal{E}^g_t(\xi)$, for all $t \in [0,T]$, and $i \in \mathbb{N}$, we use (INC) and the fact that the (Y^n, Z^n) are supersolutions to conclude

$$\begin{aligned}
&\mathcal{E}^g_s - \int_s^t g_u(\mathcal{E}^g_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \\
&\qquad\geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda^{(n)}_i \left(Y^i_s - \int_s^t g_u(Y^i_u, Z^i_u)du + \int_s^t Z^i_u dW_u \right) \\
&\qquad\geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda^{(n)}_i Y^i_t = \limsup_n \tilde{Y}^n_t = \limsup_n Y^n_t = \mathcal{E}^g_t. \quad (4.21)
\end{aligned}$$

As for the case of $s, t \in I^c$, with $s \leq t$, we approximate them both from the right by some sequences $(s^n) \subset I$ and $(t^n) \subset I$, such that $s^n \downarrow s$, $t^n \downarrow t$, $s^n \leq t^n$. For each s^n and t^n holds (4.21). Passing to the limit by using the right-continuity of $\mathcal{E}^g$ and the continuity of $-\int g(\mathcal{E}^g, \hat{Z})du + \int \hat{Z} dW$ we deduce that (4.21), holds for all $s, t \in [0,T]$ with $s \leq t$.

It remains to show admissibility of $\hat{Z}$. By means of (4.21), (4.18), and positivity of g it holds

$$\int_0^t \hat{Z}_s dW_s \geq E\left[\xi \mid \mathcal{F}_t\right] - \mathcal{E}_0. \quad (4.22)$$

Being bounded from below by a martingale, the continuous local martingale $\int \hat{Z} dW$ is by Fatou's lemma a supermartingale and thus $\hat{Z}$ is admissible. Hence, the proof under Assumptions (POS), (CON) and (INC) is completed.

The proof under (DEC) replacing (INC) only differs in the verification of (3.1). Indeed, instead of only approximating $\hat{Z}$ in the Lebesgue integral we approximate $\mathcal{E}^g(\xi)$ $P \otimes dt$-almost surely with the sequence (Y^n) as well, that is (4.20) becomes, by means of (4.19) and Fatou's lemma,

$$\mathcal{E}^g_s - \int_s^t g_u(\mathcal{E}^g_u, \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \geq \limsup_n \left(\tilde{Y}^n_s - \int_s^t g_u(Y^n_u, \tilde{Z}^n_u 1_{[0,\tau^n]}(u)) du + \int_s^t \tilde{Z}^n_u dW_u \right).$$

This entails, by monotonicity of the sequence (Y^n) and the fact that the convex combinations in $\tilde{Z}^n$ consist of elements of (Z^i) with index greater or equal than n, that we may write $-\int_s^t g_u(Y^n_u, Z^i_u) du \geq -\int_s^t g_u(Y^i_u, Z^i_u) du$ in (4.21) and this ends the proof. □

Remark 4.2. Note that the existence theorem also holds if we additionally take into account a volatility process in the stochastic integral. More precisely, consider a progressively measurable process $\sigma : \Omega \times [0,T] \to \mathbb{S}^{>0}_d$, where $\mathbb{S}^{>0}_d$ denotes the set of strictly positive definite $d \times d$ matrices and define $\mathcal{L}^\sigma$ as the set of progressively measurable processes $Z : \Omega \times [0,T] \to \mathbb{R}^{1\times d}$ such that $Z\sigma^{1/2} \in \mathcal{L}$. Analogously to the previous setting, given a generator g and a terminal condition $\xi \in L^0$, we say that $(Y,Z) \in \mathcal{S} \times \mathcal{L}^\sigma$ is a supersolution of the BSDE under volatility σ, if

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u \sigma_u^{1/2} dW_u \geq Y_t, \quad \text{and} \quad Y_T \geq \xi, \tag{4.23}$$

for all $0 \leq s \leq t \leq T$. We say that the control process is admissible, if $\int Z\sigma^{1/2} dW$ is a supermartingale, and define

$$\mathcal{A}(\xi, g, \sigma) = \{(Y,Z) \in \mathcal{S} \times \mathcal{L}^\sigma : Z \text{ is admissible and (4.23) holds}\}, \tag{4.24}$$

as well as

$$\hat{\mathcal{E}}^{g,\sigma}_t(\xi) = \operatorname{ess\,inf}\{Y_t : (Y,Z) \in \mathcal{A}(\xi, g, \sigma)\}, \quad t \in [0,T]. \tag{4.25}$$

We can formulate the following existence theorem.

Theorem 4.3. *Let g be a generator fulfilling* (POS), (LSC), (CON) *and either* (INC) *or* (DEC) *and $\xi \in L^0$ be a terminal condition, such that $\xi^- \in L^1$. If $\mathcal{A}(\xi, g, \sigma) \neq \emptyset$, then there exists a unique minimal supersolution $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g, \sigma)$. Moreover, $\mathcal{E}^{g,\sigma}(\xi)$ is the value process of the minimal supersolution, that is $(\mathcal{E}^{g,\sigma}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g, \sigma)$.*

The proof follows exactly the same scheme as the proof of Theorem 4.1 with a compactness argument in the Hilbert space $\mathcal{L}^{2,\sigma}$, the set of processes in $\mathcal{L}^\sigma$ such that $E[\int_0^T (Z_u\sigma_u^{1/2})(Z_u\sigma_u^{1/2})^\top du] < +\infty$, instead of $\mathcal{L}^2$. ♦

Theorem 4.1 ensures the existence and uniqueness of the minimal supersolution which is càdlàg. The following proposition provides a condition under which $\mathcal{E}^g(\xi)$ is in fact continuous.

Proposition 4.4. *Let g be a generator fulfilling* (POS), (LSC), (CON) *and either* (INC) *or* (DEC) *and $\xi \in L^0$ be a terminal condition, such that $\xi^- \in L^1$. Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$. Assume that for any*

$\zeta \in L^\infty(\mathcal{F}_\tau)$, $\tau \in \mathcal{T}$, there exist $Y \in \mathcal{S}$ and an admissible $Z \in \mathcal{L}$, which solve the backward stochastic differential equation

$$Y_t - \int_t^\tau g_s(Y_s, Z_s)\,ds + \int_t^\tau Z_s dW_s = \zeta, \quad \textit{for all } t \in [0,\tau].$$

Then $\mathcal{E}^g(\xi)$ is continuous.

Proof. In view of Theorem 4.1, there exists $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g, \hat{Z}) \in \mathcal{A}(\xi, g)$. Hence, $\mathcal{E}^g$ can only have downward jumps. Assume that $\mathcal{E}^g$ has a negative jump, that is $P[\tau \le T] > 0$, for the stopping time $\tau = \inf\{t > 0 : \Delta\mathcal{E}^g_t < 0\}$. We then fix m big enough such that the stopping time $\tau^m = \inf\{t > 0 : |\mathcal{E}^g_t| > m\} \wedge \tau$ satisfies $P[\{-m < \Delta\mathcal{E}^g_{\tau^m} < 0\} \cap \{\tau^m = \tau\}] > 0$. Since $\mathcal{E}^g$ is continuous on $[0,\tau[$ and $\mathcal{E}^g$ has only negative jumps, $\mathcal{E}^g_{\tau^m} \vee -m \in L^\infty(\mathcal{F}_{\tau^m})$. By assumption there exist $\bar{Y} \in \mathcal{S}$ and an admissible $\bar{Z} \in \mathcal{L}$ such that

$$\bar{Y}_s + \int_s^{\tau^m} g_u\left(\bar{Y}_u, \bar{Z}_u\right) - \int_s^{\tau^m} \bar{Z}_u dW_u = \mathcal{E}^g_{\tau^m} \vee -m, \quad \text{for all } s \in [0,\tau^m].$$

Similar to Lemma 3.1, we derive $(\bar{Y}1_{[0,\tau^m[} + \mathcal{E}^g 1_{[\tau^m,T]}, \bar{Z}1_{[0,\tau^m]} + \hat{Z}1_{]\tau^m,T]}) \in \mathcal{A}(\xi, g)$. Hence, by optimality of $\mathcal{E}^g$ in $\mathcal{A}(\xi,g)$ holds $\mathcal{E}^g \le \bar{Y}1_{[0,\tau^m[} + \mathcal{E}^g 1_{[\tau^m,T]}$. Moreover, we have

$$\mathcal{E}^g_{\tau^m -} > \mathcal{E}^g_{\tau^m} \vee -m = \bar{Y}_{\tau^m} = \bar{Y}_{\tau^m -} \quad \text{on the set } \{-m < \Delta\mathcal{E}^g_{\tau^m} < 0\} \cap \{\tau^m = \tau\}.$$

Hence, for the stopping time $\hat{\tau} = \inf\{t > 0 : \mathcal{E}^g_t > \bar{Y}_t\} \wedge \tau^m$ we deduce $P[\hat{\tau} < \tau^m] > 0$, since the processes $\mathcal{E}^g$ and $\bar{Y}$ are continuous on $[0,\tau^m[$. But then $\mathcal{E}^g \not\le \bar{Y}$ on $[0,\tau^m[$, which is a contradiction. □

Under the assumptions of Theorem 4.1, $\mathcal{E}^g$ is the value process of the minimal supersolution with a control process $\hat{Z}$ in $\mathcal{L}$ which defines a supermartingale. Next we address the question under which conditions the control process has enough integrability in order to define a true martingale, that is, when does $\hat{Z}$ belong to some $\mathcal{L}^p$, for $p \ge 1$. Defining

$$\mathcal{A}^p(\xi, g) := \{(Y,Z) \in \mathcal{A}(\xi, g) : Z \in \mathcal{L}^p\}, \tag{4.26}$$

this means that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^p(\xi, g)$. Peng [30] provides a positive answer to that question in the case where $p = 2$, the terminal condition $\xi \in L^2$ and the generator is not necessarily positive but Lipschitz. Compare also with Cheridito and Stadje [7] for supersolutions of BSDEs where the control process is in BMO, if the terminal condition is a bounded lower semicontinuous function of the Brownian motion and the generator is convex in z and Lipschitz and increasing in y. Here, we provide an answer to the case where $p = 1$ in the context of Section 3. Given a terminal condition ξ, obtaining $\mathcal{E}^g(\xi)$ as a minimal solution with a control process within $\mathcal{L}^1$ comes at two costs. Indeed, a stronger integrability condition on the terminal value is required, that is, we impose that $(E[\xi^- \,|\, \mathcal{F}_\cdot])^*_T \in L^1$. As for the second cost, $\mathcal{A}^1(\xi, g) \neq \emptyset$ is also required, which, in view of $\mathcal{A}^1(\xi, g) \subset \mathcal{A}(\xi, g)$, is also a stronger assumption.

Theorem 4.5. *Suppose that the generator g fulfills* (Pos), (Lsc), (Con) *and either* (Inc) *or* (Dec). *Let $\xi \in L^0$ be a terminal condition, such that $(E[\xi^- \,|\, \mathcal{F}_\cdot])^*_T \in L^1$. If $\mathcal{A}^1(\xi, g) \neq \emptyset$, then there exists a unique minimal supersolution $(\hat{Y}, \hat{Z}) \in \mathcal{A}^1(\xi, g)$. Moreover, $\mathcal{E}^g(\xi)$ is the value process of the minimal supersolution, that is $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^1(\xi, g)$.*

Remark 4.6. As in Section 3, note that for $(Y, Z) \in \mathcal{A}^1(\xi, g)$, the value process Y is a supermartingale with terminal value greater or equal than ξ. Moreover, we have $Y_T^* \in L^1$. Indeed, by using the decomposition (3.6), we derive $Y_t^* \leq |Y_0| + A_T + \left(\int ZdW\right)_T^*$. We further have $A_T \leq Y_0 + \int_0^T Z_s dW_s - \xi$ and thus $E\left[|A_T|\right] \leq Y_0 + E\left[\xi^-\right]$. Consequently

$$E\left[Y_T^*\right] \leq |Y_0| + E\left[\xi^-\right] + Y_0 + E\left[\left(\int ZdW\right)_T^*\right] < \infty.$$

♦

Proof (of Theorem 4.5). Since $\mathcal{A}^1(\xi, g) \subset \mathcal{A}(\xi, g)$, the assumption $\mathcal{A}^1(\xi, g) \neq \emptyset$ implies the existence of $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. We are left to show that $\hat{Z} \in \mathcal{L}^1$. Since $\mathcal{A}^1(\xi, g) \neq \emptyset$, we can suppose in the proof of Theorem 4.1 that $(Y^1, Z^1) \in \mathcal{A}^1(\xi, g)$. Since (4.3) holds for $(\mathcal{E}^g(\xi), \hat{Z})$, instead of (Y^n, Z^n), we have

$$\left(\int \hat{Z}dW\right)_T^* \leq \left|Y_0^1\right| + E\left[\xi^-\right] + \hat{A}_T + (Y^1)_T^* + \left(E\left[\xi^- \mid \mathcal{F}_\cdot\right]\right)_T^*, \tag{4.27}$$

where $0 \leq E[\hat{A}_T] \leq E[\xi] - Y_0^1$. Since $(E[\xi^- \mid \mathcal{F}_\cdot])_T^* \in L^1$, by means of Remark 4.6, the right hand side of (4.27), is in L^1. Thus, by means of the Burkholder-Davis-Gundy inequality, $\hat{Z}$ belongs to $\mathcal{L}^1$. □

4.2 Stability Results

In this section we address the stability of $\hat{\mathcal{E}}^g(\cdot)$ with respect to perturbations of the terminal condition or the generator. First we show that the functional $\hat{\mathcal{E}}_0^g$ is not only defined on the same domain as the usual expectation, but also shares some of its main properties, such as Fatou's lemma as well as a monotone convergence theorem.

Theorem 4.7. *Suppose that the generator g fulfills* (POS)*,* (LSC)*,* (CON) *and either* (INC) *or* (DEC)*. Let (ξ^n) be a sequence in L^0, such that $\xi^n \geq \eta$, for all $n \in \mathbb{N}$, where $\eta \in L^1$.*

- *Monotone convergence: If (ξ^n) is increasing P-almost surely to $\xi \in L^0$, then $\hat{\mathcal{E}}_0^g(\xi) = \lim_n \hat{\mathcal{E}}_0^g(\xi^n)$.*
- *Fatou's lemma: $\hat{\mathcal{E}}_0^g(\liminf_n \xi^n) \leq \liminf_n \hat{\mathcal{E}}_0^g(\xi^n)$.*

Proof. Monotone convergence: From Proposition 3.3 and by monotonicity, it follows that $\hat{\mathcal{E}}^g(\xi^n) \leq \hat{\mathcal{E}}^g(\xi^{n+1}) \leq \cdots \leq \hat{\mathcal{E}}^g(\xi)$. Hence, we may define $\hat{Y}_0 = \lim_n \hat{\mathcal{E}}_0^g(\xi^n)$. Note that $\hat{Y}_0 \leq \hat{\mathcal{E}}_0^g(\xi)$. If $\hat{Y}_0 = +\infty$, then also $\hat{\mathcal{E}}_0^g(\xi) = +\infty$ and there is nothing to prove. Suppose now that $\hat{Y}_0 < \infty$. This implies that $\mathcal{A}(\xi^n, g) \neq \emptyset$, for all $n \in \mathbb{N}$. Since $\xi^n \geq \eta$, Proposition 3.4 yields $(\xi^n) \subset L^1$ and $(\mathcal{E}^g(\xi^n))$ is a well-defined increasing sequence of càdlàg supermartingales. We define $Y_t = \lim_n \mathcal{E}_t^g(\xi^n)$, for all $t \in [0, T]$. Note that $Y_0 = \hat{Y}_0$. We show that Y is a càdlàg supermartingale.

To this end, note that the sequence $\left(\mathcal{E}^g(\xi^n) - \mathcal{E}^g(\xi^1)\right)$ is positive and increases to $Y - \mathcal{E}^g(\xi^1)$. Therefore monotone convergence yields

$$0 \leq E[Y_t - \mathcal{E}_t^g(\xi^1)] = \lim_n E[\mathcal{E}_t^g(\xi^n) - \mathcal{E}_t^g(\xi^1)].$$

The supermartingale property of $\mathcal{E}^g(\xi^n)$ implies that $E[\mathcal{E}_t^g(\xi^n)] \leq \mathcal{E}_0^g(\xi^n) \leq Y_0$. Furthermore, $E[\xi^1] \leq E[\mathcal{E}_t^g(\xi^1)] \leq Y_0$ and thus

$$0 \leq E[Y_t - \mathcal{E}_t^g(\xi^1)] \leq -E[\xi^1] + Y_0 < +\infty.$$

From $\mathcal{E}^g_t(\xi^1) \in L^1$, we deduce that $Y_t \in L^1$. Since $\xi = Y_T$, this implies in particular that $\xi \in L^1$. The supermartingale property follows by a similar argument. Moreover, [13, Theorem VI.18] implies that Y is indistinguishable from a càdlàg process. Hence, Y is a càdlàg supermartingale.

Theorem 4.1 provides a sequence of optimal controls (Z^n) such that $(\mathcal{E}^g(\xi^n), Z^n) \in \mathcal{A}(\xi^n, g)$, for all $n \in \mathbb{N}$. Now we apply the procedure introduced in the proof of Theorem 4.1 and obtain a candidate control process $\hat{Z}$. The only notable difference in the proof, except for the fact that Y is already càdlàg, is that, here, the sequence $(\mathcal{E}^g(\xi^n))$ is increasing instead of decreasing. Thus, the càdlàg supermartingales Y and $\mathcal{E}^g(\xi^1)$ serve as upper and lower bound, respectively. Consequently, we replace Y^1 by Y and $E[\xi^- \mid \mathcal{F}_\cdot]$ by $\mathcal{E}^g(\xi^1)$ in the key Inequality (4.3). The verification follows exactly the same argumentation as in the proof of Theorem 4.1 for both monotonicity Assumptions (INC) and (DEC). Finally, to get the admissibility of $\hat{Z}$ we denote with $(\tilde{\xi}^n)$ the sequence of convex combinations of (ξ^n) corresponding to $(\tilde{Z}^n)$. Monotonicity of the sequence (ξ^n) implies $\xi^1 \leq \tilde{\xi}^n \leq \xi$, for all $n \in \mathbb{N}$. We may and do switch to a subsequence such that $(\tilde{\xi}^n)$ is increasing as well. Now, fix an arbitrary $t \in [0,T]$. Dominated convergence implies the L^1-convergence $\lim_n E[\tilde{\xi}^n \mid \mathcal{F}_t] = E[\xi \mid \mathcal{F}_t]$. Hence, we may select a subsequence such that we have P-almost sure convergence. Similar to (4.22) this implies

$$Y_0 - \int_0^t g_u(Y_u, \hat{Z}_u)du + \int_0^t \hat{Z}_u dW_u \geq \limsup_n E\left[\tilde{\xi}^n \mid \mathcal{F}_t\right] = E\left[\xi \mid \mathcal{F}_t\right].$$

As before, this entails that $(Y, \hat{Z}) \in \mathcal{A}(\xi, g)$. Hence, from $\mathcal{A}(\xi, g) \neq \emptyset$ and $\xi^- \in L^1$ we derive by Theorem 4.1 that there exists a control process Z such that $(\mathcal{E}^g(\xi), Z) \in \mathcal{A}(\xi, g)$. In particular this yields $Y_0 = \mathcal{E}^g_0(\xi)$, that is $\lim_n \mathcal{E}^g_0(\xi^n) = \mathcal{E}^g_0(\xi)$, since otherwise $\mathcal{E}^g_0(\xi)$ were not optimal.

Fatou's lemma: The result follows by applying monotone convergence. Indeed, denote by ζ^n the random variables $\zeta^n = \inf_{k \geq n} \xi^k$. Then from $\liminf_n \xi^n = \lim_n \zeta^n$, $\zeta^n \geq \eta$, $\zeta^n \leq \xi^n$, for all $n \in \mathbb{N}$, and monotone convergence follows

$$\hat{\mathcal{E}}^g_0(\liminf_n \xi^n) = \hat{\mathcal{E}}^g_0(\lim_n \zeta^n) = \lim_n \hat{\mathcal{E}}^g_0(\zeta^n) \leq \liminf_n \hat{\mathcal{E}}^g_0(\xi^n). \qquad \square$$

Remark 4.8. An inspection of the proof of Theorem 4.7 shows that under the assumptions implying monotone convergence, if $\lim_n \hat{\mathcal{E}}^g_0(\xi^n) < +\infty$, then $\mathcal{A}(\xi, g) \neq \emptyset$ and $\mathcal{E}^g_t(\xi^n)$ converges P-almost surely to $\mathcal{E}^g_t(\xi)$, for all $t \in [0,T]$.

Similarly, given a sequence $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$ such that (Y^n) is increasing and $\lim_n Y^n_0 < \infty$, then there exists a control process $Z \in \mathcal{L}$ such that $(Y, Z) \in \mathcal{A}(\xi, g)$, where Y_t is the P-almost sure limit of (Y^n_t), for all $t \in [0,T]$. ♦

A consequence of the preceding theorem is the following result on L^1-lower semicontinuity.

Theorem 4.9. *Let g be a generator fulfilling* (POS), (LSC), (CON) *and either* (INC) *or* (DEC). *Then $\hat{\mathcal{E}}^g_0$ is L^1-lower semicontinuous.*

Proof. Let (ξ^n) be a sequence of terminal conditions, which converges in L^1 to a random variable ξ. Suppose that there exists a subsequence $(\tilde{\xi}^n) \subset (\xi^n)$ such that $(\hat{\mathcal{E}}^g_0(\tilde{\xi}^n))$ converges to some real $a < \hat{\mathcal{E}}^g_0(\xi)$. We can assume, up to another fast subsequence, that $\|\tilde{\xi}^n - \xi\|_{L^1} \leq 2^{-n}$, for all $n \in \mathbb{N}$. Consider now the sequence (ζ^n), with ζ^n given by

$$\zeta^n = \xi - \sum_{k \geq n} (\tilde{\xi}^k - \xi)^-.$$

Clearly, $\zeta^n \in L^1$ and $\zeta^n \leq \zeta^{n+1} \leq \cdots \leq \xi$. Moreover, (ζ^n) converges in L^1 to ξ, and, since it is increasing, it converges also P-almost surely. Thus, from Theorem 4.7, we get $\lim_n \hat{\mathcal{E}}_0^g(\zeta^n) = \hat{\mathcal{E}}_0^g(\xi)$. Now, $\zeta^n \leq \xi - (\tilde{\xi}^n - \xi)^- + (\tilde{\xi}^n - \xi)^+ \leq \tilde{\xi}^n$ and monotony of the functional $\hat{\mathcal{E}}_0^g$ imply $a = \lim_n \hat{\mathcal{E}}_0^g(\tilde{\xi}^n) \geq \lim_n \hat{\mathcal{E}}_0^g(\zeta^n) = \hat{\mathcal{E}}_0^g(\xi)$, which is a contradiction. Hence, $\liminf_n \hat{\mathcal{E}}_0^g(\xi^n) \geq \hat{\mathcal{E}}_0^g(\xi)$. □

The preceding results allows to derive a dual representation, by means of the Fenchel-Moreau theorem, of the functional $\hat{\mathcal{E}}^g(\cdot)$ at time zero.

Corollary 4.10. *Let g be a generator fulfilling* (POS), (LSC) *and either* (INC) *or* (DEC). *Assume that g is jointly convex in y and z. Then, either $\hat{\mathcal{E}}_0^g \equiv +\infty$ or*

$$\hat{\mathcal{E}}_0^g(\xi) = \mathcal{E}_0^g(\xi) = \sup_{\nu \in L_+^\infty} \left\{ E[\nu\xi] - \left(\hat{\mathcal{E}}_0^g\right)^*(\nu) \right\}, \quad \xi \in L^1, \tag{4.28}$$

for the conjugate $(\hat{\mathcal{E}}_0^g)^*(\nu) = \sup_{\xi \in L^1} \left\{ E[\nu\xi] - \hat{\mathcal{E}}_0^g(\xi) \right\}$, where $\nu \in L^\infty$.

Proof. Since $\hat{\mathcal{E}}_0^g > -\infty$ on L^1, either $\hat{\mathcal{E}}_0^g \equiv +\infty$ or $\hat{\mathcal{E}}_0^g$ is proper. In the latter case, in view of Proposition 3.3 and Theorem 4.9, the function $\hat{\mathcal{E}}_0^g$ is convex and $\sigma(L^1, L^\infty)$-lower semicontinuous on L^1. Hence, the Fenchel Moreau theorem yields the dual representation (4.28). That the domain of $(\hat{\mathcal{E}}_0^g)^*$ is concentrated on L_+^∞ follows from the monotonicity of $\hat{\mathcal{E}}_0^g$, see Proposition 3.3. □

Remark 4.11. Notice that, if the generator in Corollary 4.10 does not depend on y, then by Item 5 of Proposition 3.3 the operator $\hat{\mathcal{E}}_0^g(\cdot)$ is translation invariant. Therefore, it is a lower semicontinuous, convex risk measure and the Representation (4.28) corresponds to the robust representation of lower semicontinuous, convex risk measures; see Föllmer and Schied [21]. ♦

Under additional integrability assumptions on the terminal condition we may also formulate stability results for supersolutions in the set $\mathcal{A}^1(\xi, g)$ introduced in (4.26).

Theorem 4.12. *Suppose that the generator g fulfills* (POS), (LSC), (CON) *and either* (DEC) *or* (INC). *Let (ξ^n) be a sequence in L^0, such that $\xi^n \geq \eta$, for all $n \in \mathbb{N}$, where $(E[\eta \mid \mathcal{F}.])_T^* \in L^1$.*

- *Suppose (ξ^n) is increasing P-almost surely to $\xi \in L^0$ and $\mathcal{A}^1(\xi, g) \neq \emptyset$. Then $\mathcal{E}_t^g(\xi) = \lim_n \mathcal{E}_t^g(\xi^n)$, P-almost surely, for all $t \in [0, T]$.*
- *Suppose $\mathcal{A}^1(\liminf_n \xi^n, g) \neq \emptyset$. Then $\mathcal{E}_t^g(\liminf_n \xi^n) \leq \liminf_n \mathcal{E}_t^g(\xi^n)$, P-almost surely, for all $t \in [0, T]$.*

We omit the proof of the preceding theorem, as it is a simple adaptation of the proofs of Theorems 4.5 and 4.7. Note that Theorem 4.12 is a weaker version of Theorem 4.7. Indeed, here, given a sequence (ξ^n) increasing to ξ, we need to assume that $\mathcal{A}^1(\xi, g)$ is not empty. The underlying reason being the lack of knowledge whether the limit process Y, defined in the proof of Theorem 4.7, fulfills $Y_T^* \in L^1$.

The theorem above allows to state the following result on $\mathcal{L}^1$-lower semicontinuity of $\hat{\mathcal{E}}^g$. Its proof is virtually the same as the proof of Theorem 4.9.

Theorem 4.13. *Suppose that the generator g fulfills* (POS), (LSC), (CON) *and either* (DEC) *or* (INC). *Then $\xi \mapsto \hat{\mathcal{E}}_0^g(\xi)$ is $\mathcal{L}^1$-lower semicontinuous on its domain, that is on*

$$\left\{ \xi \in L^0 : (E[\xi^- \mid \mathcal{F}.])_T^* \in L^1 \text{ and } \mathcal{A}^1(\xi, g) \neq \emptyset \right\}. \tag{4.29}$$

We conclude this section with a theorem on monotone stability with respect to the generator.

Theorem 4.14. *Let $\xi \in L^0$ be a terminal condition, such that $\xi^- \in L^1$, and let (g^n) be an increasing sequence of generators, which converge pointwise to a generator g. Suppose that each generator fulfills* (POS), (LSC), (CON) *and either* (INC) *or* (DEC). *Then $\lim_n \hat{\mathcal{E}}_0^{g^n}(\xi) = \hat{\mathcal{E}}_0^g(\xi)$. If, in addition, $\lim_n \hat{\mathcal{E}}_0^{g^n}(\xi) < \infty$, then $\mathcal{A}(\xi, g) \neq \emptyset$ and $\mathcal{E}_t^{g^n}(\xi)$ converges P-almost surely to $\mathcal{E}_t^g(\xi)$, for all $t \in [0, T]$.*

Remark 4.15. Under additional assumptions on the generators one can prove Fatou type stability results for a $P \otimes dt$-almost sure converging sequence of generators, see Gerdes et al. [22] for details. ♦

Proof. Note that from Proposition 3.3, we have $\hat{\mathcal{E}}^{g^n}(\xi) \leq \hat{\mathcal{E}}^{g^{n+1}}(\xi) \leq \cdots \leq \hat{\mathcal{E}}^g(\xi)$. Hence, we may set $\hat{Y}_0 = \lim_n \hat{\mathcal{E}}_0^{g^n}(\xi)$. If $\hat{Y}_0 = \infty$, then also $\hat{\mathcal{E}}_0^g(\xi) = \infty$ and we are done. Suppose that $\hat{Y}_0 < \infty$. By the same arguments as in the proof of Theorem 4.7, we constructa a càdlàg supermartingale Y. With the same procedure as in Theorem 4.7, we construct the candidate $\hat{Z}$. It remains to show $(Y, \hat{Z}) \in \mathcal{A}(\xi, g)$. However, this can be done similarly as in the proof of Theorem 4.1. We only show how to obtain the analogue of (4.21). Note first that the pointwise convergence of the generators implies that $(g^k(Y, \hat{Z}))$ converges $P \otimes dt$-almost surely to $g(Y, \hat{Z})$. Hence, Fatou's lemma yields

$$Y_s - \int_s^t g_u(Y_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \geq \limsup_k \left(Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \right). \tag{4.30}$$

As in the previous proof, we use the expression in the bracket on the right hand side to obtain

$$\begin{aligned} Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left(Y_s^i - \int_s^t g_u^k(Y_u^i, Z_u^i)du + \int_s^t Z_u^i dW_u \right). \end{aligned}$$

Since on the right hand side we consider the $\limsup$ with respect to n and k being fixed for the moment, we may assume $k \leq n$, which entails by monotonicity of the sequence of generators

$$\begin{aligned} Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \sum_{i=n}^{M^{(n)}} \lambda_i^{(n)} \left(Y_s^i - \int_s^t g_u^i(Y_u^i, Z_u^i)du + \int_s^t Z_u^i dW_u \right). \end{aligned}$$

From here, we obtain as before $Y_s - \int_s^t g_u^k(Y_u, \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u \geq Y_t$, where the right hand side does not depend on k anymore. Combined with (4.30), this yields the analogue of (4.21). □

4.3 Non positive generators

In this section we extend our results to generators that are not necessarily positive. Using some measure change, the positivity assumption on the generator g can be relaxed to a linear bound below. This leads to optimal solutions under P, where the admissibility is required with respect to the related equivalent probability measure. More precisely, we say in the following that a generator g is

(LB) linearly bounded from below, if there exist adapted measurable $\mathbb{R}^{1\times d}$ and $\mathbb{R}$-valued processes a and b, respectively, such that $g(y,z) \geq az^\top + b$, for all $y,z \in \mathbb{R}\times\mathbb{R}^{1\times d}$. Furthermore, $\int_0^t b_s ds \in L^1(P^a)$, for all $t\in[0,T]$ and

$$\frac{dP^a}{dP} = \mathcal{E}\left(\int adW\right)_T,$$

defines an equivalent probability measure P^a.

Example 4.16. For instance, given a generator g, assume that there exists a generator $\hat{g}$ independent of y fulfilling (CON) and such that $g \geq \hat{g}$. Then, there exists an $\mathbb{R}^{1\times d}$-valued adapted measurable process a such that $g\left(y,z\right) \geq az^\top - \hat{g}^*(a)$, for all $y,z\in\mathbb{R}\times\mathbb{R}^{1\times d}$, where $\hat{g}^*$ denotes the convex conjugate of $\hat{g}$.$\Diamond$

In the following, we say that Z is a-admissible, if $\int ZdW^a$ is a P^a-supermartingale, where $W^a = (W^1 - \int a^1 ds, \cdots, W^d - \int a^d ds)^\top$ is the respective Brownian motion under P^a. We are interested in the sets

$$\mathcal{A}^a\left(\xi,g\right) = \left\{(Y,Z)\in\mathcal{S}\times\mathcal{L} : Z \text{ is } a\text{-admissible and (3.1) holds}\right\}, \tag{4.31}$$

and define the random process

$$\hat{\mathcal{E}}_t^{g,a}(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0\left(\mathcal{F}_t\right) : (Y,Z)\in\mathcal{A}^a\left(\xi,g\right)\right\}, \quad t\in[0,T]. \tag{4.32}$$

The analogue of Theorem 4.1 is given as follows

Theorem 4.17. *Let g be a generator fulfilling* (LB), (LSC), (CON) *and either* (INC) *or* (DEC) *and $\xi\in L^0$ be a terminal condition, such that $\xi^-\in L^1(P^a)$. If $\mathcal{A}^a(\xi,g)\neq\emptyset$, then there exists a unique minimal supersolution $(\hat{Y},\hat{Z})\in\mathcal{A}^a(\xi,g)$. Moreover, $\mathcal{E}^g\left(\xi\right)$ is the value process of the minimal supersolution, that is $(\mathcal{E}^g\left(\xi\right),\hat{Z})\in\mathcal{A}^a(\xi,g)$.*

The analogues of Theorem 4.7 and Theorem 4.9 read as follows.

Theorem 4.18. *Suppose that the generator g fulfills* (LB), (LSC), (CON) *and either* (INC) *or* (DEC). *Let (ξ^n) be a sequence in L^0, such that $\xi^n\geq\eta$, for all $n\in\mathbb{N}$, where $\eta\in L^1(P^a)$.*

- *Monotone convergence: If (ξ^n) is increasing P-almost surely to $\xi\in L^0$, then $\hat{\mathcal{E}}_0^{g,a}(\xi) = \lim_n \hat{\mathcal{E}}_0^{g,a}(\xi^n)$.*
- *Fatou's lemma: $\hat{\mathcal{E}}_0^{g,a}(\liminf_n \xi^n) \leq \liminf_n \hat{\mathcal{E}}_0^{g,a}(\xi^n)$.*

In particular, $\hat{\mathcal{E}}_0^{g,a}$ is $L^1(P^a)$-lower semicontinuous.

We only prove the first theorem.

Proof (of Theorem 4.17). In the setting of Section 4.1, given a positive generator $\bar{g}$ and a random variable ζ, let us denote by $\mathcal{A}\left(\zeta,\bar{g},W^a\right)$ the set defined in (3.3) to indicate the dependence of this set on the Brownian motion W^a and the respective probability measure P^a. Let us now define the generator $\bar{g}$ as

$$\bar{g}\left(y,z\right) = g\left(y + \int_0^{\cdot} b_s ds, z\right) - az^\top - b, \quad \text{for all } (y,z)\in\mathbb{R}\times\mathbb{R}^{1\times d}. \tag{4.33}$$

By Assumption (LB), this generator fulfills (POS), (LSC), (CON) and either (INC) or (DEC). Since $\int ZdW^a$ is a P^a-supermartingale, a simple inspection shows that the affine transformation $\bar{Y} = Y - \int bds$ and $\bar{Z} = Z$ yields a one-to-one relation between $\mathcal{A}^a(\xi,g)$ and $\mathcal{A}(\xi - \int_0^T b_s ds, \bar{g}, W^a)$. Hence, the assumptions of Theorem 4.1 are fulfilled for $\bar{g}$ and $\mathcal{A}(\xi - \int_0^T b_s ds, \bar{g}, W^a)$, and thus its application ends the proof. $\square$

Remark 4.19. Note that if $(E^a[(\xi - \int_0^T b_s ds)^- \mid \mathcal{F}_\cdot])_T^* \in L^1(P^a)$, then Theorem 4.5 applies in the same way, that is, under the assumptions of Theorem 4.17, if

$$\mathcal{A}^{1,a}(\xi, g) := \left\{(Y, Z) \in \mathcal{A}^a(\xi, g) : Z \in \mathcal{L}^1(P^a)\right\} \neq \emptyset,$$

then $\mathcal{E}^{g,a}(\xi)$ is the value process of the minimal supersolution with unique control process $Z \in \mathcal{L}^1(P^a)$.♦

References

[1] S. Ankirchner, P. Imkeller, and G. Dos Reis. Classical and variational differentiability of BSDEs with quadratic growth. *Electronic Journal of Probability*, 12(53):1418–1453, 2007.

[2] P. Artzner, F. Delbaen, J. M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9:203–228, 1999.

[3] M. T. Barlow and P. Protter. On convergence of semimartingales. In *Séminaire de Probabilités, XXIV, 1988/89*, volume 1426 of *Lecture Notes in Math.*, pages 188–193. Springer, Berlin, 1990.

[4] J. Bion-Nadal. Time consistent dynamic risk processes. *Stochastic Processes and their Applications*, 119(2):633–654, 2009.

[5] P. Briand and F. Confortola. BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces. *Stochastic Processes and their Applications*, 118(5):818–838, 2008.

[6] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields*, 141(3–4):543–567, 2008.

[7] P. Cheridito and M. Stadje. Existence, minimality and approximation of solutions to BSDEs with convex drivers. *Stochastic Processes and their Applications*, 122(4):1540–1565, 2012.

[8] P. Cheridito, F. Delbaen, and M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. *Electronic Journal of Probability*, 11(3):57–106, 2006.

[9] F. Delbaen. The structure of m-stable sets and in particular of the set of risk neutral measures. In *In Memoriam Paul-André Meyer–Seminaire de Probabilités XXXIX*, pages 215–258. Springer, 2006.

[10] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(3):463–520, 1994.

[11] F. Delbaen and W. Schachermayer. A compactness principle for bounded sequences of martingales with applications. In *Proceedings of the Seminar of Stochastic Analysis, Random Fields and Applications, Progress in Probability*, pages 137–173. Birkhauser, 1996.

[12] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with superquadratic growth. *Probability Theory and Related Fields*, 150(1-2):145–192, 2011.

[13] C. Dellacherie and P.-A. Meyer. *Probabilities and Potential. B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982. Theory of martingales, Translated from the French by J. P. Wilson.

[14] Dudley. Wiener functionals as Itô integrals. *Annals of Probability*, 5(1):140–141, 1977.

[15] D. Duffie and L. G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–94, 1992.

[16] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control Optimization*, 33(1):29–66, 1995.

[17] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez. Reflected solutions of backward SDEs, and related obstacle problems for PDEs. *The Annals of Probability*, 25(2):702–737, 1997.

[18] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 1(1):1–71, January 1997.

[19] H. Föllmer and I. Penner. Convex risk measures and the dynamics of their penalty functions. *Statistics & Decisions*, 24(1):61–96, 2006.

[20] H. Föllmer and A. Schied. Convex measures of risk and trading constraint. *Finance and Stochastics*, 6(4):429–447, 2002.

[21] H. Föllmer and A. Schied. *Stochastic Finance. An Introduction in Discrete Time*. de Gruyter Studies in Mathematics. Walter de Gruyter, Berlin, New York, 2 edition, 2004.

[22] H. Gerdes, G. Heyne, and M. Kupper. Stability of minimal supersolutions of BSDEs. *Preprint*, 2012.

[23] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215–260, 1981.

[24] G. Heyne. *Essays on minimal supersolutions of BSDEs and on cross hedging in incomplete markets*. PhD thesis, Humboldt Universität zu Berlin, 2012.

[25] G. Heyne, M. Kupper, and C. Mainberger. Minimal supersolutions of BSDEs with lower semicontinuous generators. *accepted for publication in Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 2012.

[26] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics)*. Springer, August 2004.

[27] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, 28(2):558–602, 2000.

[28] F. Maccheroni, M. Marinacci, and A. Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, November 2006.

[29] S. Peng. Backward SDE and related g-expectation. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Research Notes in Mathematics Series*, pages 141–159. Longman, Harlow, 1997.

[30] S. Peng. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyer's type. *Probability Theory and Related Fields*, 113(4):473–499, 1999.

[31] S. Peng. G-expectation, G-Brownian motion and related stochatic calculus of Itô type. *Stochastic Analysis and Applications*, volume 2 of Abel Symp.:541–567, 2007.

[32] S. Peng. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Processes and their Applications*, 12:2223–2253, 2008.

[33] S. Peng and M. Xu. The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 41(3):605–630, 2005.

[34] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, 2004.

[35] A. Réveillac. Weak martingale representation for continuous markov processes and application to quadratic growth BSDEs. *Preprint*, 2011.

[36] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.