On the Solution of Optimization and Variational Problems with Imperfect Information

Uday V. Shanbhag (with Hao Jiang (@Illinois) and Hesam Ahmadi (@PSU))

Harold and Inge Marcus Department of Industrial and Manufacturing Engineering
Pennsylvania State University
University Park, PA

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A prototypical misspecified* convex program where $\theta^* \in \mathbb{R}^m$ is misspecified:

$$
\begin{align*}
C(\theta^*) & \quad \text{minimize} \quad f(x, \theta^*) \\
\end{align*}
$$

Generally, $\theta^*$ captures problem characteristics that may require estimation.

- Parameters of cost/price functions
- Efficiencies
- Representation of uncertainty

Generally, this is part of the model building process.

- Traditionally, a dichotomy in the roles of statisticians and optimizers

1. Statisticians Learn – (Build model, estimate parameters)
2. Optimizers Search – (Use model/parameters to obtain solution)

- Increasingly, the serial nature cannot persist.

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*This is parametric misspecification (as opposed to model misspecification)
Offline learning I

- One avenue lies in collecting observations a priori
- Learning problem $\mathcal{L}_\theta$ unaffected by the computational problem $\mathcal{C}(\theta^*)$:

\[
\begin{array}{c}
\mathcal{L}_\theta \\
\text{minimize} \\
\theta \in \Theta \\
g(\theta)
\end{array}
\]

Concerns:

- **Exact** solutions generally unavailable in **finite** time; solution error can be bounded in expected-value sense (at best) in stochastic regimes
- Premature termination of learning process leads to $\hat{\theta}$; Error cascades into computational problem;
  \[\hat{x} \in \text{SOL}(\mathcal{C}(\hat{\theta})).\]
- Unclear how to develop\(^a\) implementable scheme that produces $x^*$:
  - (First-order) schemes that produce $x^*$ and $\theta^*$ asymptotically
  - Non-asymptotic error bounds

\(^a\) Note that schemes that produce approximations are available based on Lipschitzian properties
An example I

\[ c(x; \theta^*) \triangleq \frac{1}{2} \theta_1 x + \theta_2 x^2 \]
An example II

\[ c(x; \theta^*) \]

\[ c(x_1; \theta^*) + \xi_1 \quad c(x_2; \theta^*) + \xi_2 \quad c(x_M; \theta^*) + \xi_M \]
An example III

$$L_\theta$$

$$\theta^* \in \arg\min_{\theta \in \Theta} \sum_{\ell=1}^{M} \left\| \frac{1}{2} \theta_1 x_\ell + \theta_2 x_\ell^2 - \hat{c}(x_\ell; \theta^*) \right\|^2$$

$$c(x; \theta^*)$$

$$c(x_1; \theta^*) + \xi_1$$

$$c(x_2; \theta^*) + \xi_2$$

$$c(x_3; \theta^*) + \xi_3$$

$$c(x_M; \theta^*) + \xi_M$$
Consider the following static stochastic program

\[ \min_{x \in X} \mathbb{E}[f(x, \xi_{\theta^*}(\omega))], \quad (C_{\theta^*}) \]

where \( f : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R} \), \( \xi_{\theta^*} : \Omega \rightarrow \mathbb{R}^d \) and \((\Omega, \mathcal{F}, \mathbb{P}_{\theta^*})\) represents the probability space.

Traditionally, the parameters of this distribution are estimated a priori (by MLE approaches for instance). Often a challenging problem (such as covariance selection)
Misspecified production planning problems I

▶ The production planner solves the following problem:

\[
\min_{x_{fi} \geq 0} \sum_{f=1}^{N} \sum_{i=1}^{W} c_{fi}(x_{fi})
\]

subject to \( x_{fi} \leq \text{cap}_{fi} \), for all \( f, i \).

\[
\sum_{i=1}^{N} x_{fi} = d_i.
\]

(1)

▶ Machine type \( f \)'s production cost at node \( i \) \( c_{fi}^{(l)}(x_{fi}^{(l)}) \) at time \( l \), \( l = 1, \ldots, T \):

\[
c_{fi}^{(l)}(x_{fi}^{(l)}) = d_{fi}(x_{fi}^{(l)})^2 + h_{fi}x_{fi}^{(l)} + \xi_{fi}^{(l)}
\]

▶ The planner will solve the following problem to estimate \( d_{fi} \) and \( h_{fi} \):

\[
\min_{\{d_{fi}, h_{fi}, i\} \in \Theta} \sum_{l=1}^{T} \sum_{f=1}^{N} \sum_{i=1}^{W} (d_{fi}(x_{fi}^{(l)})^2 + h_{fi}x_{fi}^{(l)} - c_{fi}^{(l)}(x_{fi}^{(l)}))^2.
\]
Our focus is on general purpose algorithms that \textit{jointly} generate sequences \( \{x_k\} \) and \( \{\theta_k\} \) with the following goals:

\[
\lim_{k \to \infty} x_k = x^* \quad \text{and} \quad \lim_{k \to \infty} \theta_k = \theta^* \quad \text{(Global convergence)}
\]

\[
\|f(x_K, \theta_K) - f(x^*, \theta^*)\| \leq \mathcal{O}(h(K)), \quad \text{(Rate statements)}
\]

where \( h(K) \) specifies the rate.
A serial approach

1. Compute a solution $\tilde{\theta}$ to $(L_\theta)$
2. Use solution to solve $(C(\tilde{\theta}))$

Challenges:

- Given the stage-wise nature, step 1. needs to provide accurate/exact $\tilde{\theta}$ in finite time; possible for small problems;
- In stochastic regimes, solution bounds available in expected-value sense:
  \[
  \mathbb{E}[\|\theta_K - \theta^*\|^2] \leq O(1/K).
  \]
- In fact, unless the learning problem is solvable via a finite termination algorithm, asymptotic statements are unavailable.
A complementarity approach

- A direct variational approach: under convexity assumptions, equilibrium conditions are given by $\text{VI}(Z, H)$ where

$$H(z) \triangleq \left( \begin{array}{c} F(x, \theta) \\ \nabla_\theta g(\theta) \end{array} \right)$$

and $Z \triangleq X \times \Theta$.

Challenges:

- Problem rarely monotone and low-complexity first-order projection/stochastic approximation schemes cannot accommodate such problems.
Research questions

- First-order schemes available for solution of deterministic/stochastic convex optimization and monotone variational problems
- Can we develop analogous schemes that guarantee global/a.s. convergence†
- Can rate statements be provided for such schemes:
  - Are the original rates preserved?
  - What is the price of learning in terms of the modification/degradation in rates?

† not immediate since problems can be viewed as non-monotone VIs/SVIs.
Part I: Deterministic problems:
  - Gradient methods for smooth/nonsmooth and strongly convex/convex optimization
  - Extragradient and regularization methods for monotone variational inequality problems

Part II: Stochastic problems:
  - Stochastic approximation schemes for strongly convex/convex stochastic optimization with stochastic learning problems
  - Regularized stochastic approximation for monotone stochastic variational inequality problems with stochastic learning problems
Literature Review

Static decision-making problems with perfect information

- Optimization: convex programming [BNO03], integer programming [NW99], stochastic programming [BL97]
- Variational inequality problems [FP03a]

Learning

- Linear and nonlinear regression, support vector machines (SVMs), etc. [HTF01]

Joint schemes for related problems:
- Adaptive control [AW94], Iterative learning (tracking) control [Moo93]
- Bandit problems [Git89], regret problems [Zin03]
- Relatively less on joint schemes focusing on stylized problems in revenue management [CHdMK06, HKZ, CHdMK12]
Misspecified deterministic optimization

Consider the static misspecified convex optimization problem \((C(\theta^*))\):

\[
\min_{x \in X} f(x, \theta^*), \quad (C(\theta^*))
\]

where \(x \in \mathbb{R}^n, f : X \times \Theta \to \mathbb{R}\) is a convex function in \(x\) for every \(\theta \in \Theta \subseteq \mathbb{R}^m\). Suppose \(\theta^*\) denotes the solution to a convex learning problem denoted by \((L)\):

\[
\min_{\theta \in \Theta} g(\theta), \quad (L)
\]

where \(g : \mathbb{R}^m \to \mathbb{R}\) is a convex function in \(\theta\) and is defined on a closed and convex set \(\Theta\).
A joint gradient algorithm

Algorithm 1 (Joint gradient scheme)

Given $x_0 \in X$ and $\theta_0 \in \Theta$ and sequences $\gamma_{f,k}, \gamma_{g,k},$

$$x_{k+1} := \Pi_X (x_k - \gamma_{f,k} \nabla_x f(x_k, \theta_k)), \quad \forall k \geq 0.$$  \hspace{1cm} (Opt($\theta_k$))

$$\theta_{k+1} := \Pi_\Theta (\theta_k - \gamma_{g,k} \nabla_\theta g(\theta_k)), \quad \forall k \geq 0.$$  \hspace{1cm} (Learn)
Assumptions

Assumption 1
The function $f(x, \theta)$ is continuously differentiable in $x$ for all $\theta \in \Theta$ and function $g$ is continuously differentiable in $\theta$.

Assumption 2
The gradient map $\nabla_x f(x; \theta)$ is Lipschitz continuous in $x$ with constant $G_{f,x}$ uniformly over $\theta \in \Theta$ or

$$\| \nabla_x f(x_1, \theta) - \nabla_x f(x_2, \theta) \| \leq G_{f,x} \| x_1 - x_2 \|, \quad \forall x_1, x_2 \in X, \quad \forall \theta \in \Theta.$$ 

Additionally, the gradient map $\nabla_\theta g$ is Lipschitz continuous in $\theta$ with constant $G_g$.

Assumption 3
Let $\{ \gamma_{f,k} \}$ and $\{ \gamma_{g,k} \}$ be diminishing nonnegative sequences chosen such that $\sum_{k=1}^{\infty} \gamma_{f,k} = \infty$, $\sum_{k=1}^{\infty} \gamma_{f,k}^2 < \infty$, $\sum_{k=1}^{\infty} \gamma_{g,k} = \infty$, and $\sum_{k=1}^{\infty} \gamma_{g,k}^2 < \infty$. 
Assumption 4
The function $f$ is strongly convex in $x$ with constant $\eta_f$ for all $\theta \in \Theta$ and the function $g$ is strongly convex with constant $\eta_g$.

Assumption 5
The gradient $\nabla_x f(x^*, \theta)$ is Lipschitz continuous in $\theta$ with constant $L_\theta$.

Proposition 1 (Rate analysis in strongly convex regimes)
Let Assumptions 1, 2, 4 and 5 hold. In addition, assume that $\gamma_f$ and $\gamma_g$ are chosen such that $\gamma_f \leq \min(2\eta_f/G^2_{f,x}, 1/L_\theta)$ and $\gamma_g \leq 2/G_g$. Let $\{x_k, \theta_k\}$ be the sequence generated by Algorithm 1. Then for every $k \geq 0$, we have the following:

$$\|x_{k+1} - x^*\| \leq q_x^{k+1} \|x_0 - x^*\| + kq_\theta q^{k} \|\theta_0 - \theta^*\|,$$

where $q_x \triangleq (1 + \gamma_f^2 G^2_{f,x} - 2\gamma_f \eta_f)^{1/2}$, $q_\theta \triangleq \gamma_f L_\theta$, $q_g \triangleq (1 + \gamma_g^2 G^2_g - 2\gamma_g \eta_g)^{1/2}$, and $q \triangleq \max(q_x, q_g)$. 
**Remark:** Notably, learning leads to a degradation in the convergence rate from the standard **linear** rate to a **sub-linear** rate. Furthermore, it is easily seen that when we have access to the true $\theta^*$, the original rate may be recovered.

‡

**Figure 1:** Strongly convex problems and learning: Constant steplength (l) and Diminishing steplength (r)
Figure 2: Strongly convex optimization and learning: Impact on rate (l) and empirical vs. theor. rate (r)

† We provide some numerics on a small production planning problem with 5 plants with capacity and ramping requirements. We assume that either cost is misspecified (Opt) or demand is misspecified (VIs).
Misspecified convex optimization I

Assumption 6

*The function* $f$ *is convex in* $x$ *with constant* $\eta_f$ *for all* $\theta \in \Theta$ *and the function* $g$ *is strongly convex with constant* $\eta_g$.

Assumption 7

(a) *The sets* $X$ *and* $\Theta$ *are compact and* $\sup_{x \in X} \|x\| \leq C$, *where* $C$ *is a constant.*

(b) *The gradient map* $\nabla_x f(x; \theta)$ *is uniformly Lipschitz continuous in* $\theta$ *with constant* $G_{f, \theta}$:

$$\|\nabla_x f(x, \theta_1) - \nabla_x f(x, \theta_2)\| \leq G_{f, \theta}\|\theta_1 - \theta_2\|, \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$$

Assumption 8

*There exists a constant* $L_{f, \theta}$ *such that*

$$|f(x, \theta_1) - f(x, \theta_2)| \leq L_{f, \theta}\|\theta_1 - \theta_2\|, \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$$
Proposition 2 (Constant steplength scheme with averaging)

Let Assumptions 1, 2, 6, 7 and 8 hold and stepsizes $\gamma_{f,k}$ and $\gamma_{g,k}$ be fixed at constants $\gamma_f$ and $\gamma_g$ so that $0 < \gamma_g < 2/G_g$ and $0 < \gamma_f \leq 1/G_{f,x}$. Let the sequence $\{x_k, \theta_k\}$ be generated by Algorithm 1 and suppose $\bar{x}_k$ is defined as

$$\bar{x}_k \triangleq \frac{1}{k} \sum_{i=0}^{k-1} x_{i+1}.$$ 

Then the following hold:

(i) In addition, if $a_x = \frac{\|x_0 - x^*\|^2}{2\gamma_f}$, $a_\theta \triangleq \|\theta_0 - \theta^*\|$, and $b_\theta \triangleq \frac{CG_{f,\theta}}{1-q_g}$, then the following holds:

$$|f(\bar{x}_K, \theta_K) - f(x^*, \theta^*)| \leq \frac{a_x}{K} + a_\theta \left( \frac{b_\theta}{K} + L_{f,\theta} q_g^K \right).$$

(ii) $\lim_{k \to \infty} f(\bar{x}_k, \theta_k) = f(x^*, \theta^*).$
Remarks:

- Unlike in the case of strongly convex optimization, there is no degradation in the standard rate of convergence in function values which is $\mathcal{O}(1/K)$.
- Contribution from learning is given by

\[ \|\theta_0 - \theta^*\| \left( L_{f,\theta} q_g^K + \frac{b_\theta}{K} \right). \]

- Some intuition:
  - The first term arises from the effort to learn the correct $\theta^*$
  - The second term is an interaction term between $x$ and $\theta$ through $L_{f,\theta}$ and is mitigated by averaging
  - Both terms are scaled by $\|\theta_0 - \theta^*\|$.
  - The overall rate does not degrade (but gets modified)
Figure 3: Convex optimization and strongly convex learning: Impact on rate (l) and empirical vs. theor. (r)
Nonsmooth convex optimization I

Assumption 9

The function $g$ is continuously differentiable in $\theta$, strongly convex, and the gradient map $\nabla_\theta g(\theta)$ is Lipschitz continuous in $\theta$ with constant $G_g$.

Assumption 10 (Subgradient boundedness)

There exists an $M > 0$ such that $\|d_k\| \leq M$ for all $d_k \in \partial f(x_k, \theta_k)$ and for all $\theta_k \in \Theta$.

Assumption 11

There exists a constant $L_{f,\theta}$ such that

$$|f(x, \theta_1) - f(x, \theta_2)| \leq L_{f,\theta} \|\theta_1 - \theta_2\| \quad \forall \theta_1, \theta_2 \in \Theta, x \in X.$$

We consider the following subgradient-based analog of Algorithm 1:

Algorithm 2 (Joint subgradient scheme)

Given an $x_0 \in X$ and a $\theta_0 \in \Theta$ and sequences $\{\gamma_f,k, \gamma_g,k\}$, then

$$x_{k+1} := \Pi_X (x_k - \gamma_f,k d_k), \quad \forall k \geq 0, \quad \text{ (nsOpt}(\theta_k))$$

$$\theta_{k+1} := \Pi_\Theta (\theta_k - \gamma_g,k \nabla_\theta g(\theta_k)), \quad \forall k \geq 0, \quad \text{ (Learn)}$$

where $d_k \in \partial f(x_k, \theta_k)$. 
Proposition 3 (Rate analysis with averaging)

Let Assumptions 9, 10, and 11 hold. Let $\gamma_{g,k}$ be fixed at $\gamma_g$ such that $0 < \gamma_g < 2/G_g$. Consider the sequence $\{x_k, \theta_k\}$ generated by Algorithm 2 and $\bar{x}_k \triangleq \frac{\sum_{i=0}^{k} \gamma_{f,i} x_i}{\sum_{i=0}^{k} \gamma_{f,i}}$. Then the following hold:

(i) If $\gamma_{f,k}$ is defined based on Assumption 3 with $\gamma_{f,0} \leq 2\eta_f / \eta_{f,x}$ and $\gamma_g \leq 2/G_g$, then

$$\lim_{k \to \infty} |f(\bar{x}_k, \theta_k) - f(x^*, \theta^*)| = 0.$$ 

(ii) Suppose Algorithm 2 is to be terminated after $K$ iterations and $\gamma_f$ (the optimal constant steplength) is defined as $\gamma_{f,K} = \frac{\|x_0 - x^*\|}{M \sqrt{K+1}}$, then

$$|f(\bar{x}_K, \theta_K) - f(x^*, \theta^*)| \leq \frac{d_x}{\sqrt{K+1}} + d_{\theta} \left( L_{f,\theta} q_g^K + \frac{c_{\theta}}{(K+1)} \right),$$

where $d_x = M\|x_0 - x^*\|$, $d_{\theta} = \|\theta_0 - \theta^*\|$, and $c_{\theta} = 2L_{f,\theta} / (1 - q_g)$. 
Remark: Standard subgradient methods for convex optimization display a convergence rate of $O(1/\sqrt{K})$ in function value [BV04] using optimal constant steplength [SDR09]

- Joint scheme shows no degradation in the rate, not even in a constant factor sense.
- Modification in the rate is given by

$$
\|\theta_0 - \theta^*\| \left( L_{r,\theta} q_g^K + \frac{b_\theta}{K} \right).
$$

- Identical to the smooth case
Nonsmooth convex optimization IV

![Graph showing error in function value vs iteration. Theoretical error is represented by a red line, and empirical error is represented by a green dashed line. The x-axis represents iteration, and the y-axis represents error in function value. The graph illustrates a decrease in error as the iteration increases.]
The misspecified optimization problem is now generalized to a variational inequality problem:

\[(y - x)^T F(x; \theta^*) \geq 0, \quad \forall y \in X. \quad (\mathcal{V}(\theta^*))\]

Assumption 12

(a) The function \( g \) is differentiable, strongly convex with constant \( \eta_g \), and Lipschitz continuous in gradient with constant \( G_g \).

(b) The map \( F \) is monotone in \( x \) and uniformly Lipschitz continuous in \( x \) and \( \theta \) with constants \( L_{F,x} \) and \( L_{F,\theta} \), respectively:

\[
\|F(x_1; \theta) - F(x_2; \theta)\| \leq L_{F,x} \|x_1 - x_2\| \quad \forall x_1, x_2 \in X, \quad \forall \theta \in \Theta,
\]

\[
\|F(x, \theta_1) - F(x, \theta_2)\| \leq L_{F,\theta} \|\theta_1 - \theta_2\| \quad \forall \theta_1, \theta_2 \in \Theta, \quad \forall x \in X.
\]
Algorithm 3 (A joint extragradient scheme)

Given an $x_0 \in X$ and a $\theta_0 \in \Theta$ and a steplength $\tau$,

\[
\begin{align*}
z_{k+1} &:= \Pi_X(x_k - \tau F(x_k; \theta_k)) & \forall k > 0, & \text{(Extra}_x(\theta_k)) \\
x_{k+1} &:= \Pi_X(x_k - \tau F(z_{k+1}; \theta_k)) & \forall k > 0, & \text{(Extra}_z(\theta_k)) \\
\theta_{k+1} &:= \Pi_{\Theta}(\theta_k - \gamma_g \nabla_{\theta} g(\theta_k)) & \forall k > 0. & \text{(Learn)}
\end{align*}
\]

Theorem 1 (Convergence of extragradient scheme)

Let Assumption 12 holds and $\Theta$ is bounded. In addition, assume that stepsize $\gamma_{g,k}$ is fixed at $\gamma_g$, where $\gamma_g \leq \frac{2}{G_g}$. Let $\{x_k, \theta_k\}$ be the sequence generated by Algorithm 3 with

\[
\tau^2 \leq \frac{1}{L_{F,X}^2 + L_{F,\theta} \|\theta_0 - \theta^*\|}.
\]

Then, $\{x_k\}$ converges to a point in $X^*$ and $\{\theta_k\}$ converges to $\theta^* \in \Theta$ as $k \to \infty$. 
Remark:

- Standard extragradient methods require that \( \tau \leq \frac{1}{L_{f,x}} \) (cf. [FP03b]).
- This variant requires that
  \[
  \tau \leq \sqrt{\frac{1}{L_{f,x}^2 + L_{f,0} \| \theta_0 - \theta^* \|}}.
  \]
- When \( \theta_0 = \theta^* \), we recover the original result.
Iteratively (Tikhonov) regularized schemes

- Tikhonov regularization techniques [Tik63, TA76, FP03b] have proved useful in solving monotone variational inequality problems.
- Specifically, such techniques construct a sequence \( \{x_k\} \) where
  \[
  x_k = \Pi_X (x_k - \gamma_k (F(x_k) + \epsilon_k x_k)), \quad \forall k \geq 0
  \]
  implying that \( x_k \in \text{SOL}(X, F+\epsilon_k I) \), where \( \{\epsilon_k\} \to 0 \) and \( \{x_k\} \to x^* \in X^* \).
- Challenge: obtaining \( x_k \) requires solving a strongly monotone VI exactly (or with increasing accuracy) at every step
- An alternative lies in using \textit{iterative} Tikhonov regularization where a \textit{projected gradient} step is taken at every step [Pol87, KS10]
  \[
  x_{k+1} := \Pi_X (x_k - \gamma_k (F(x_k) + \epsilon_k x_k)), \quad \forall k \geq 0.
  \]
  Under suitable assumptions of \( \{\gamma_k, \epsilon_k\} \), convergence can be recovered.
- We consider an extension of this scheme to the misspecified regime.

**Algorithm 4 (A regularized projection scheme)**

\[\text{Given an } x_0 \in X \text{ and } \theta_0 \in \Theta \text{ and sequences } \{\gamma_{f,k}\} \text{ and } \{\epsilon_k\}, \]

\[
x_{k+1} := \Pi_X (x_k - \gamma_{f,k} (F(x_k, \theta_k) + \epsilon_k x_k)), \quad \forall k > 0, \quad \text{(Var}(\theta_k, \epsilon_k))
\]

\[
\theta_{k+1} := \Pi_{\Theta} (\theta_k - \gamma_{g,k} \nabla_{\theta} g(\theta_k)), \quad \forall k > 0. \quad \text{(Learn)}
\]
Iteratively (Tikhonov) regularized schemes II

In our analysis, we consider two auxiliary sequences \{x^t_k\} and \{z^t_k\}, defined as follows:

\[
x^t_k := \Pi_X(x^t_k - \gamma_{f,k}(F(x^t_k, \theta_k) + \epsilon_k x^t_k)) \quad \forall k > 0, \quad \text{(Tik}(\theta_k))
\]
\[
z^t_k := \Pi_X(z^t_k - \gamma_{f,k}(F(z^t_k, \theta^*) + \epsilon_k z^t_k)) \quad \forall k > 0. \quad \text{(Tik}(\theta^*))
\]

- \{z^t_k\} is the Tikhonov trajectory under perfect information (\theta^* is known)
- \{x^t_k\} is the Tikhonov trajectory under belief \theta_k
- Proof of convergence shows that \|x_k - x^t_k\| \to 0 as \(k \to \infty\) and \|x^t_k - z^t_k\| \to 0 as \(k \to \infty\).

Lemma 1

Let Assumptions 12, 13 and 14(d) hold. Suppose \(x^t_k\) and \(x^t_{k-1}\) are defined by Tik(\(\theta_k\)) and Tik(\(\theta_{k-1}\)) respectively. Then, we have that \(\|x^t_k - x^t_{k-1}\|\) can be bounded as follows:

\[
\|x^t_k - x^t_{k-1}\| \leq \frac{L_{F,\theta} q_g^{k-1} C_g}{\epsilon_k} + \frac{M}{\epsilon_k} |\epsilon_{k-1} - \epsilon_k|,
\]

where \(q_g \triangleq \sqrt{1 - 2\gamma_g \eta_g + \gamma_g^2 G_g^2}\), \(C_g \triangleq \|\theta_0 - \theta^*\|(1 + q_g)\), and \(M\) is the constant defined in Assumption 13.
Assumption 13

*The set $X$ is compact and $\sup_{x \in X} \| x \| \leq M$, where $M$ is a constant.*

Assumption 14

*The following hold:*

(a) $0 < \gamma_{f,k} \leq \frac{\epsilon_k}{(L_{F,x} + \epsilon_k)^2} \leq \frac{\epsilon_0}{L_{F,x}^2}$ for all $k$;

(b) $\gamma_{f,k} \epsilon_k < 1$ and $\sum_{k=1}^{\infty} \gamma_{f,k} \epsilon_k = \infty$;

(c) $\lim_{k \to \infty} \frac{|\epsilon_{k-1} - \epsilon_k|}{\gamma_{f,k} \epsilon_k^2} = 0$;

(d) $\gamma_{g,k} \triangleq \gamma_g$ such that $\gamma_g \leq 2\eta_g / G_g^2$ and $\lim_{k \to \infty} \frac{q_g^{k-1}}{\gamma_{f,k} \epsilon_k^2} = 0$, where $q_g \triangleq \sqrt{1 - 2\gamma_{g,k} \eta_g + \gamma_{g,k}^2 G_g^2}$.

**Theorem 2 (Convergence of regularized scheme)**

Let Assumptions 12, 13 and 14 hold. Consider the sequence $\{x_k, \theta_k\}$ generated by Algorithm 4. Then, $\{x_k\}$ converges to $x^*$ as $k \to \infty$, where $x^*$ denotes the least-norm solution of $X^*$ and $\{\theta_k\}$ converges to $\theta^* \in \Theta$. 
Introduction of uncertainty I

- **Computational problem:** We consider the stochastic generalization of optimization/variational inequality problems.
- Specifically, such a problem requires an \( x^* \in X \) such that
  \[
  (x - x^*)^T \mathbb{E}[F(x^*; \theta^*, \xi(\omega))] \geq 0, \quad \forall x \in X, \quad (P_x(\theta^*))
  \]
  where \( \xi : \Omega \rightarrow \mathbb{R}^d, F : X \times \mathbb{R}^d \rightarrow \mathbb{R}^n, X \subseteq \mathbb{R}^n \), and \((\Omega, \mathcal{F}, \mathbb{P})\) denotes the probability space.
- **Learning problem:** The vector \( \theta^* \) lies in the solution set of \((P_{\theta})\):
  \[
  \min_{\theta \in \Theta} g(\theta), \text{ where } g(\theta) \triangleq \mathbb{E}[g(\theta; \eta)]. \quad (P_{\theta})
  \]
Algorithm 5 (Coupled SA schemes for stochastic opt. problems)

Step 0. Given \( x_0 \in X, \theta_0 \in \Theta \) and sequences \( \{\gamma_k, x, \gamma_k, \theta\} \), \( k := 0 \)

Step 1.

\[
x^{k+1} := \Pi_X \left( x^k - \gamma_{k,x} (\nabla_x f(x^k; \theta^k) + w^k) \right), \quad k \geq 0
\]

\[
\theta^{k+1} := \Pi_\Theta \left( \theta^k - \gamma_{k,\theta} (\nabla_\theta g(\theta^k) + v^k) \right), \quad k \geq 0
\]

\( w^k \triangleq \nabla_x f(x^k; \theta^k, \xi^k) - \nabla_x f(x^k; \theta^k) \) and \( v^k \triangleq \nabla_\theta g(\theta^k; \eta^k) - \nabla_\theta g(\theta^k) \).

Step 2. If \( k > K \), stop; else \( k : k + 1 \), go to Step 1.
Assumptions

Assumption 1 (Problem properties, A1-1)
Suppose the following hold:

(i) For every $\theta \in \Theta$, $f(x; \theta)$ is strongly convex ($\mu_x$) and continuously differentiable with Lipschitz continuous gradients ($L_x$) in $x$.

(ii) For every $x \in X$, the gradient $\nabla_x f(x; \theta)$ is Lipschitz continuous in $\theta$ with constant $L_\theta$.

(iii) The function $g(\theta)$ is strongly convex and continuously differentiable with Lipschitz continuous gradients in $\theta$ with convexity constant $\mu_\theta$ and Lipschitz constant $C_\theta$, respectively.

Assumption 2 (Steplength requirements, A2-1)
Let $\{\gamma_{k,x}\}$ and $\{\gamma_{k,\theta}\}$ be chosen such that $\sum_{k=0}^\infty \gamma_{k,x} = \infty$, $\sum_{k=0}^\infty \gamma_{k,x}^2 < \infty$ and $\gamma_{k,\theta} = \gamma_{k,x} L_\theta^2 / (\mu_x \mu_\theta)$.

Assumption 3 (A3)
\[\mathbb{E}[w^k | \mathcal{F}_k] = 0 \text{ and } \mathbb{E}[v^k | \mathcal{F}_k] = 0 \text{ a.s. for all } k. \text{ Furthermore, } \mathbb{E}[\|w^k\|^2 | \mathcal{F}_k] \leq \nu_x^2 \text{ and } \mathbb{E}[\|v^k\|^2 | \mathcal{F}_k] \leq \nu_\theta^2 \text{ a.s. for all } k.\]

\[\text{\[}^8\text{We define a new probability space } (Z, \mathcal{F}, \mathbb{P}), \text{ where } Z \triangleq \Omega \times \Lambda, \mathcal{F} \triangleq \mathcal{F}_x \times \mathcal{F}_\theta \text{ and } \mathbb{P} \triangleq \mathbb{P}_x \times \mathbb{P}_\theta. \text{ We use } \mathcal{F}_k \text{ to denote the sigma-field generated by the initial points } (x^0, \theta^0) \text{ and errors } (w^l, v^l) \text{ for } l = 0, 1, \ldots, k - 1, \text{ i.e., } \mathcal{F}_0 = \{(x^0, \theta^0)\} \text{ and } \mathcal{F}_k = \{(x^0, \theta^0), (w^l, v^l), l = 0, 1, \ldots, k - 1\} \text{ for } k \geq 1. \text{ We make the following assumptions on the filtration and errors.}\]
Main results

Proposition 4 (Almost-sure convergence under strong convexity of $f$)

Suppose (A1-1), (A2-1) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, $x^k \to x^*$ and $\theta^k \to \theta^*$ a.s. as $k \to \infty$, where $x^*$ denotes the unique solution to $(P_x(\theta^*))$.

- Proof relies on super-martingale convergence theorem
- Surprising aspects:
  - The steplength sequences run on the same timescale; merely scaled variants
  - The overall variational problem in $(x, \theta)$ is not necessarily monotone but can be solved; what does this suggest regard the solution of more general complementarity/equilibrium/variational problems

¶ No available schemes for solving non-monotone stochastic variational inequality problems
Weakening strong convexity of \((P_x)\)

Assumption 4 (A1-2)
Suppose the following holds in addition to (A1-1 (ii)) and (A1-1 (iii)) For every \(\theta \in \Theta\), \(f(x; \theta)\) is convex and continuously differentiable with Lipschitz continuous gradients in \(x\) with Lipschitz constant \(L_x\).

Furthermore, we make the following assumptions on the steplength sequences employed in the algorithm.

Assumption 5 (A2-2)
Let \(\{\gamma_{k,x}\}, \{\gamma_{k,\theta}\}\) and some constant \(\tau \in (0, 1)\) be chosen such that \(\sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty\) and \(\sum_{k=0}^{\infty} \gamma_{k,\theta}^2 < \infty\), \(\sum_{k=0}^{\infty} \gamma_{k,x} = \infty\) and \(\sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty\), \(\beta_k = \frac{\gamma_{k,x}^{\tau}}{2\gamma_{k,\theta} \mu_{\theta}} \downarrow 0\) as \(k \rightarrow \infty\).
Proceeding as in the previous result, we present a convergence result under these weakened conditions.

Theorem 2 (Almost-sure convergence under convexity of $f$)

Suppose (A1-2), (A2-2) and (A3) hold. Suppose $X$ is bounded and the solution set $X^*$ of $(P_x(\theta^*))$ is nonempty. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, $\theta^k \to \theta^*$ a.s. as $k \to \infty$, and $x^k$ converges to a random point in $X^*$ a.s. as $k \to \infty$.

Notably, in merely convex regimes, $\gamma_{k,x}$ and $\gamma_{k,\theta}$ are run at differing timescales; specifically, $\gamma_{k,x} \to 0$ at a faster rate than $\gamma_{k,\theta} \to 0$. 
Proposition 5 (Rate estimates for strongly convex $f$)
Suppose (A1-1) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then, the following hold:

$$
\mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{Q_\theta(\lambda_\theta)}{k} \quad \text{and} \quad \mathbb{E}[\|x^k - x^*\|^2] \leq \frac{Q_x(\lambda_x)}{k},
$$

where $Q_\theta(\lambda_\theta) \triangleq \max \left\{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \right\}$,

$$
Q_x(\lambda_x) \triangleq \max \left\{ \lambda_x^2 \tilde{M}^2 (\mu_x \lambda_x - 1)^{-1}, \mathbb{E}[\|x^1 - x^*\|^2] \right\},
$$

and $\tilde{M} \triangleq \sqrt{M^2 + \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x \lambda_x}}$.

---

^aSuppose $\gamma_{x,k} = \lambda_x/k$ and $\gamma_{\theta,k} = \lambda_\theta/k$ with $\lambda_x > 1/\mu_x$ and $\lambda_\theta > 1/(2\mu_\theta)$. Let $\mathbb{E}[\|\nabla f(x^k; \theta^k) + w^k\|^2] \leq M^2$ and $\mathbb{E}[\|\nabla g(\theta^k) + v^k\|^2] \leq M_\theta^2$ for all $x^k \in X$ and $\theta^k \in \Theta$.

- Under strong convexity, optimization and learning recovers optimal rate of SA
- Naturally, when $\theta_1 = \theta^*$, we recover the original optimization result
Theorem 3 (Rate estimates under convexity of $f$)
Suppose (A1-2) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Then the following holds for $1 \leq i \leq k$:

$$\mathbb{E}[|f(\tilde{x}_{i,k}; \theta^k) - f(x^*; \theta^*)|] \leq \frac{\sqrt{Q_\theta(\lambda_\theta)}D_\theta + C_{i,k}\sqrt{B_k}}{\sqrt{k}},$$

where $C_{i,k} = \frac{k}{k-i+1}$ and $B_k = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k))(M^2 + M_x^2)$.

---

Averaging in stochastic convex optimization leads to $O(1/\sqrt{k})$

Averaging with learning leads to bound given loosely by $O\left(\sqrt{\ln(k)/\sqrt{k}}\right)$.

Degradation in learning is $O\left(\sqrt{\ln(k)}\right)$.
Constant steplength error bounds

In many multiagent systems, constant steplengths (or gain sequences) are convenient; can one quantify these errors?

Proposition 6

Suppose (A3) holds. Suppose $\gamma_{\theta,k} = \gamma_{x,k} := \gamma$. Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M^2_x$ and $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$ for all $x^k \in X$. Suppose $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$ and $a_k \triangleq \mathbb{E}[A_k]$. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5. Suppose (A1-1) holds. Then, the following holds:

$$ \limsup_{k \to \infty} a_k \leq \frac{1}{2\mu_x} \gamma M^2 + \frac{L^2_{\theta}}{2\mu_x^2} \frac{\gamma \nu^2_{\theta}}{2\mu_{\theta} - \gamma C^2_{\theta}}. $$

Suppose (A1-2) holds. Then, the following holds:

$$ \limsup_{k \to \infty} |\mathbb{E}[f(x^k; \theta^k) - f(x^*; \theta^*)]| \leq \frac{1}{2} \gamma M^2 + \frac{1}{2} \gamma^{1-\tau} M^2_x + \frac{\gamma \nu^2_{\theta}}{4\mu_{\theta} - 2\gamma C^2_{\theta}} + D_{\theta} \sqrt{\frac{\gamma \nu^2_{\theta}}{2\mu_{\theta} - \gamma C^2_{\theta}}}. $$

where $0 < \tau < 1$.

- Utility of this result; we’ve set $\gamma_x = \gamma_{\theta}$; But we may optimize this error bound in the choices of steplengths.
## Summary of rate statements

<table>
<thead>
<tr>
<th></th>
<th>Computation</th>
<th>Computation &amp; Learning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Det. Strongly convex/diff.</td>
<td>Linear</td>
<td>Sublinear</td>
</tr>
<tr>
<td>Det. convex/diff.</td>
<td>$O(1/K)$</td>
<td>$O(1/K + q_g^K)$</td>
</tr>
<tr>
<td>Det. convex/nonsmooth.</td>
<td>$O(1/\sqrt{K})$</td>
<td>$O(1/\sqrt{K}) + O(1/K + q_g^K)$</td>
</tr>
<tr>
<td>Stoch. Strongly convex</td>
<td>$O\left(\frac{1}{k}\right)$</td>
<td>$O\left(\frac{1}{k}\right)$</td>
</tr>
<tr>
<td>Stoch. Convex</td>
<td>$O\left(\frac{1}{\sqrt{k}}\right)$</td>
<td>$O\left(\frac{\sqrt{\ln(k)}}{\sqrt{k}}\right)$</td>
</tr>
</tbody>
</table>
Algorithm 6 (Coupled SA schemes for Stochastic variational probs.)

Step 0. Given $x_0 \in X, \theta_0 \in \Theta$ and sequences $\{\gamma_{k,x}, \gamma_{k,\theta}\}, k := 0$

Step 1.

$$x^{k+1} := \Pi_X \left( x^k - \gamma_{k,x} (F(x^k; \theta^k) + w^k) \right)$$ \hspace{1cm} \text{(Comp}_k \text{)}

$$\theta^{k+1} := \Pi_{\Theta} \left( \theta^k - \gamma_{k,\theta} (G(\theta^k) + v^k) \right),$$ \hspace{1cm} \text{(Learn}_k \text{)}

where $w^k \triangleq F(x^k; \theta^k, \xi^k) - F(x^k; \theta^k)$ and $v^k \triangleq G(\theta^k; \eta^k) - G(\theta^k)$.

Step 2. If $k > K$, stop; else $k := k + 1$, go to Step 1.

We begin by stating an assumption similar to (A1-1) on the mapping $F$.

Assumption 6 (A1-3)

(Identical to A1-1) with $\nabla f(x; \theta)$ replaced by $F(x; \theta)$
Proposition 7 (Almost-sure convergence under strongly monotone $F$)

Suppose (A1-3), (A2-1) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 6. Then, $x^k \to x^*$ a.s. and $\theta^k \to \theta^*$ a.s. as $k \to \infty$, where $x^*$ is the unique solution to $\text{VI}(X, F(\bullet; \theta^*))$.

- Result is similar to that for strongly convex problems
Main results II

Algorithm 7 (Coupled regularized SA schemes for stochastic VIs)

**Step 0.** Given \( x_0 \in X, \theta_0 \in \Theta \) and sequences \( \{\gamma_k, x, \gamma_k, \theta, \epsilon_k\} \), \( k := 0 \)

**Step 1.**

\[
x^{k+1} := \Pi_X \left( x^k - \gamma_k, x \left(F(x^k; \theta^k) + \epsilon_k x^k \right) + w^k \right)
\]

\( (\text{Comp}_k) \)

\[
\theta^{k+1} := \Pi_{\Theta} \left( \theta^k - \gamma_k, \theta \left(G(\theta^k) + v^k \right) \right)
\]

\( (\text{Learn}_k) \)

where \( w^k \triangleq F(x^k; \theta^k, \xi^k) - F(x^k; \theta^k) \) and \( v^k \triangleq G(\theta^k; \eta^k) - G(\theta^k) \).

**Step 2.** If \( k > K \), stop; else \( k : k + 1 \), go to Step 1.

- Unlike in optimization, we need to employ a Tikhonov regularizer, inspired by past work [KNS13]
Assumptions

The following assumptions will be made on both the decision variable and parameter.

Assumption 7 (A1-4)
(Similar to A1-3)
We also make the following assumptions on the steplength sequences employed in the algorithm.

Assumption 8 (A2-3)
Let \( \{\gamma_k, x\}, \{\gamma_k, \theta\}, \{\epsilon_k\} \) and some constant \( \tau \in (0, 1) \) be chosen such that:

(i) \[ \sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty \text{ and } \sum_{k=0}^{\infty} \gamma_{k,\theta}^2 < \infty, \]

(ii) \[ \sum_{k=0}^{\infty} \gamma_{k,x} \epsilon_k^k = \infty \text{ and } \sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty, \]

(iii) \[ \beta_k = \frac{\gamma_{k,x}}{2\gamma_{k,\theta} \mu_{\theta}} \downarrow 0 \text{ as } k \to 0. \]

(iv) \[ \sum_{k=0}^{\infty} \frac{(\epsilon_{k-1} - \epsilon_k)}{\epsilon_k} < \infty. \]
Main results

Theorem 4
Suppose (A1-4), (A2-3) and (A3) hold. Suppose $X$ is bounded and the solution set $X^*$ of $\text{VI}(X, F(\cdot, \theta^*))$ is nonempty. Let $\{x^k, \theta^k\}$ be computed via Algorithm 7. Then, $\theta^k \to \theta^*$ a.s. as $k \to \infty$, and $x^k$ converges to the least norm solution in $X^*$ a.s. as $k \to \infty$.

- Again, $\gamma_k, x$ and $\gamma_k, \theta$ are decreased at different rates
- Unlike in the optimization setting, we recover the least-norm solution
In the strongly monotone regime, we may recover the optimal rate of SA.

Without strong monotonicity, one avenue lies in averaging and working in a weak sharp regime; specifically, we assume that $\text{VI}(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$ possesses the MPS property, which is introduced in the following lemma.

**Lemma 3**

[Mar93] Let $H : X \to \mathbb{R}^n$ be a mapping that is monotone over the compact polyhedral set $X$. Let $X^*$ be the solution set of $\text{VI}(X, H)$ and there exists a positive number $\alpha$ s.t.

$$(x - x^*)^T H(x^*) \geq \alpha \text{dist}(x, X^*), \quad \forall x \in X, \quad \forall x^* \in X^*,$$

where $\text{dist}(x, X^*) \triangleq \min_{x^* \in X^*} \|x - x^*\|$. 


Theorem 5 (Rate estimates under monotonicity of $F$)

Suppose (A1-4) and (A3) hold. Let $\{x^k, \theta^k\}$ be computed via Algorithm 6. Then there exists a positive number $\alpha$ such that for $1 \leq i \leq k$:

$$\mathbb{E} \left[ \alpha \text{ dist}(\tilde{x}_{i,k}, X^*) \right] \leq C_{i,k} \sqrt{\frac{B_k}{k}},$$

where $C_{i,k} = \frac{k}{k-i+1}$ and $B_k = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k))(M^2 + M_X^2)$.

---

- Akin to merely convex regimes, averaging allows for prescribing rates
- Degradation from learning is $O\left(\sqrt{\ln(k)}\right)$.

---

$^a$Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_X^2$, $\mathbb{E}[\|F(x^k; \theta^k) + w^k\|^2] \leq M^2$ and $\mathbb{E}[\|G(\theta^k) + v^k\|^2] \leq M_\theta^2$ for all $x^k \in X$ and $\theta^k \in \Theta$. Suppose $X$ is a compact polyhedral set, the solution set $X^*$ of $\text{VI}(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$ is nonempty, and $x^*$ is a point in $X^*$. Suppose $\text{VI}(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$ possesses the MPS property.

$^b$For $1 \leq i, t \leq k$, we define $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^{k} \gamma_{x,s}}$, $\tilde{x}_{i,k} \triangleq \sum_{t=i}^{k} v_t x^t$ and $D_X \triangleq \max_{x \in X} \|x - x^1\|$. Suppose for $1 \leq t \leq k$, $\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k)}{(M^2 + M_X^2)k}}$, where $Q_\theta(\lambda_\theta) \triangleq \max \big\{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \big\}$, and $\gamma_{\theta,k} = \lambda_\theta / k$ with $\lambda_\theta > 1 / (2\mu_\theta)$.

---

$^\|$ If the $\text{VI}(X, H)$ possesses the minimum principle sufficiency (MPS) property
Proposition 8
Suppose (A3) holds. Suppose $\gamma_{\theta,k} = \gamma_{x,k} := \gamma_{x}$. Suppose $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ and $\mathbb{E}[F(x^k; \theta^k) + w^k\|^2] \leq M^2$ for all $x^k \in X$. Suppose $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$ and $a_k \triangleq \mathbb{E}[A_k]$. Suppose $X$ is a compact polyhedral set, the solution set $X^*$ of $\text{VI}(X, F(\bullet, \theta^*))$ is nonempty, and $x^*$ is a point in $X^*$. Suppose $\text{VI}(X, F(\bullet, \theta^*))$ possesses the MPS property. Let $\{x^k, \theta^k\}$ be computed via Algorithm 5.

Suppose (A1-3) holds. Then, the following holds:

$$\limsup_{k \to \infty} a_k \leq \frac{1}{2\mu_x} \gamma M^2 + \frac{L^2_\theta}{2\mu_x} \frac{\gamma \nu^2_\theta}{2\mu_\theta - \gamma C^2_\theta};$$

Suppose (A1-4) holds. Then, there exists a positive number $\alpha$ such that:

$$\limsup_{k \to \infty} \mathbb{E}[\text{dist}(x^k, X^*)] \leq \frac{1}{\alpha} \left[ \frac{1}{2} \gamma M^2 + \frac{1}{2} \gamma^{1-\tau} M_x^2 + \frac{\gamma^\tau \nu^2_\theta L^2_\theta}{4\mu_\theta - 2\gamma C^2_\theta} \right],$$

where $0 < \tau < 1$. 

Constant steplength errors
Diminishing steplength

Table 1: Distributed scheme for learning $x^*$ and $\theta^*$ in a stochastic regime: $\xi \sim U[-\theta^*/2, \theta^*/2]$

<table>
<thead>
<tr>
<th>N</th>
<th>W</th>
<th>$\frac{\mathbb{E}[|x^K-x^<em>|]}{1+|x^</em>|}$</th>
<th>$\frac{\text{ERR}}{1+|x^*|}$</th>
<th>$\frac{\mathbb{E}[|\theta^K-\theta^<em>|]}{1+|\theta^</em>|}$</th>
<th>$\text{ERR} \frac{1}{1+|\theta^*|}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>$7.4 \times 10^{-2}$</td>
<td>$1.2 \times 10^{10}$</td>
<td>$4.7 \times 10^{-2}$</td>
<td>$5.0 \times 10^4$</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>$6.5 \times 10^{-2}$</td>
<td>$2.3 \times 10^{10}$</td>
<td>$3.7 \times 10^{-2}$</td>
<td>$5.1 \times 10^4$</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>$5.8 \times 10^{-2}$</td>
<td>$3.8 \times 10^{10}$</td>
<td>$2.9 \times 10^{-2}$</td>
<td>$5.1 \times 10^4$</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>$5.8 \times 10^{-2}$</td>
<td>$6.9 \times 10^{10}$</td>
<td>$2.2 \times 10^{-2}$</td>
<td>$6.4 \times 10^4$</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>$6.7 \times 10^{-2}$</td>
<td>$1.1 \times 10^{11}$</td>
<td>$1.9 \times 10^{-2}$</td>
<td>$7.5 \times 10^4$</td>
</tr>
</tbody>
</table>

▶ $\gamma_k, x = 10/k$ and $\gamma_k, \theta = 10/k$.

▶ $K = 10000$.

▶ ERR : theoretical error in Proportion 5.
Averaging

Table 2: Distributed scheme for learning $x^*$ and $\theta^*$ in a stochastic regime: $\xi \sim U[-\theta^*/2, \theta^*/2]$

| N  | W | $\mathbb{E}[|f(\tilde{x}_1, K; \theta^K) - z^*|]$ | $\text{ERR} \frac{1 + \|x^*\|}{1 + \|z^*\|}$ | $\gamma x$ |
|----|----|---------------------------------|----------------------------------|-------|
| 10 | 2  | $1.2 \times 10^{-1}$             | $1.7 \times 10^5$                | 68    |
| 10 | 4  | $1.9 \times 10^{-1}$             | $2.1 \times 10^5$                | 92    |
| 10 | 6  | $1.1 \times 10^{-1}$             | $1.2 \times 10^5$                | 127   |
| 10 | 8  | $1.2 \times 10^{-1}$             | $1.5 \times 10^5$                | 152   |
| 10 | 10 | $1.4 \times 10^{-1}$             | $2.4 \times 10^5$                | 161   |

- $\gamma_{K, \theta} = 10/K$, $z^* = f(x^*; \theta^*)$.
- $K = 10000$.
- $\text{ERR}$ : theoretical error in Theorem 3.
Figure 4: Computing $x^*$ and learning $\theta^*$ ($\xi \sim U[-\theta^*/2, \theta^*/2]$, $N = 5$, $W = 5$)

- $\gamma_{k,x} = k^{-0.8}$, $\gamma_{k,\theta} = 10/k$, $z^* = f(x^*; \theta^*)$.
- $K = 10000$.
- ERR: theoretical error in Theorem ??.
Concluding remarks

A broad framework for resolving misspecified stochastic optimization/variational problems:

▶ Asymptotics for gradient/subgradient/extragradient/iterative regularization schemes for deterministic problems
▶ (a.s.) Asymptotics for stochastic approximation (and regularized counterparts) for stochastic problems
▶ Rate statements for gradient/subgradient schemes with quantification of impact; Similar statements for mean-squared error for stochastic approximation schemes

Key findings:

▶ Natural extensions of gradient-type schemes are provably convergent
▶ Recover optimal rates upto constant factor modifications in some regimes; degradation in other regimes.
▶ **Seemingly non-monotone problems in full-space can be solved via first order schemes with modest rate degradation at worst**

Ongoing work:

▶ Misspecified Markov Decision Processes (as an alternative to Q-learning) where transition matrices need to be learnt
▶ Consensus (distributed optimization) under imperfect information

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