First exit times of solutions of non-linear stochastic differential equations driven by symmetric Lévy processes with α-stable components

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Abstract

We study the exit problem of solutions of the stochastic differential equation $dX_t^\varepsilon = -U'(X_t^\varepsilon)\,dt + \varepsilon\,dL_t$ from bounded or unbounded intervals which contain the unique asymptotically stable critical point of the deterministic dynamical system $\dot{Y}_t = -U'(Y_t)$. The process L is composed of a standard Brownian motion and a symmetric α-stable Lévy process. Using probabilistic estimates we show that in the small noise limit $\varepsilon \to 0$, the exit time of X^ε from an interval is an exponentially distributed random variable and determine its expected value. Due to the heavy-tail nature of the α-stable component of L, the results differ strongly from the well known case in which the deterministic dynamical system undergoes purely Gaussian perturbations.

Keywords: Lévy process, Lévy flight, first exit, exit time law, α–stable process, Kramers' law, infinitely divisible distribution, extreme events.

AMS Subject classification: 60E07, 60F10, 60G40, 60G51, 60G52 60H10, 60J75, 60K40, 86A17

Introduction

The study of dynamical systems subject to small random perturbations keeps receiving much attention both in the physical and the mathematical literature. In the simplest (one-dimensional) setting, systems of this type find the following mathematical formulation. Consider the ordinary differential equation $\dot{Y}_t = -U'(Y_t)$, $Y_0 \in [-b, a]$, $a, b > 0$, where U is a potential function. Assume that $U(0) = 0$ and that 0 is the unique asymptotically stable point of the deterministic dynamical system associated with the equation, that means that for any starting point x in $[-b, a]$, the deterministic trajectory tends to 0: $Y_t(x) \to 0$ as $t \to \infty$.

Now perturb the deterministic dynamical system with some small random noise, that is, consider the solutions of the stochastic differential equation

$$X_t^\varepsilon = x - \int_0^t U'(X_s^\varepsilon)\,ds + \varepsilon\eta_t, \qquad (*)$$

where η is a random process, and the noise intensity parameter ε is small compared to the other parameters of the system (ε is close to 0). Under certain conditions on U and η, for example under the assumptions that U' is Lipschitz and η is a semimartingale, the solution of the equation $(*)$ is well defined. In case η is a standard Brownian motion, the dynamical system is said to be perturbed by white noise. If η is an Ornstein-Uhlenbeck process, the terminology of 'red noise' perturbation has been used. The literature also knows perturbations by the so-called shot noises, fractional Gaussian noises, or Lévy noises.

The stochastic dynamics of systems perturbed by white noise, which belong to the large class of diffusions driven by the Brownian motion, in the small noise limit, i.e. for $\varepsilon \to 0$, has received a great deal of attention for decades and is particularly well understood. The pioneering papers on this topic are

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[PAV33, Eyr35, Kra40]. Later on it was studied in [Fri74, Sch80, Wil82, Day83, BM92, KM96, FW98, Kol00, BEGK02, BGK02], as well as in many other papers.

One of the main results in this field is concerned with the time it takes for the diffusion to exit a neighborhood of a local attractor. Due to the fact that Kramers' pioneering paper was one of the first to derive heuristically some properties of an *exit law* of this type, in particular in the physical literature it is often named Kramers' law. Stated in modern language, it says that the expected exit time is exponentially large in ε^{-2} and the growth rate can be interpreted as the height of the potential barrier to be overcome to leave the local attractor neighborhood (see section 1 for a rigorous formulation).

White noise perturbations, however, are not always appropriate to interpret real data in a reasonable way. This is the case for example if the nature of the underlying random perturbation process has to model abrupt pulses or *extreme events*. A more natural mathematical framework for these phenomena takes into account other than purely Brownian perturbations. In particular infinitely divisible Lévy perturbations with jumps enter the stage.

The physical papers by P. Ditlevsen [Dit99a, Dit99b] motivating our research stipulate more general noise sources of the type alluded to. They originate in simple physical concepts serving to interpret paleoclimatic data. In fact, paleoclimatic records from the Greenland ice-core show that the climate of the last glacial period experienced rapid transitions between cold basic glacial periods and several warmer interstadials (the so-called Dansgaard–Oeschger events). Those records are given by the concentration of certain oxygen, hydrogen or calcium isotopes in the annual layers of the ice-core extending over several hundred millennia in the past. They can be used to reconstruct the global Earth temperature for the time span for which the records are available. The calcium signal has the highest — almost annual — temporal resolution and provides the most conclusive information about the statistics of the Dansgaard–Oeschger warming events. They start with a very rapid warming of the North Atlantic region of about 5–10°C within at most a few decades. The warming is followed by a plateau phase with slow cooling extending over several centuries, ending with an equally abrupt drop to basic glacial conditions. The nature of these events is not clear, and several conceptual explanations have been proposed. One line of arguments invokes instabilities in the North Atlantic thermohaline circulation as a causal mechanism for these abrupt climate changes, and for the millennial time scale between jumps. A different reasoning starts from the idea that the coupled atmosphere-ocean system in the tropics possesses several meta-stable states, and claims global teleconnections that trigger changes in the North Atlantic thermohaline circulation. In [GR01], the effect of stochastic resonance was brought into play in order to explain the observed random periodicity of the Dansgaard–Oeschger events.

As for many climate phenomena, due to the non-linearity of the system the physical background is highly complex. To understand some basic features, simple low-dimensional models may be used. For the phenomena under consideration, in this spirit one may hope to recover important aspects of the statistical properties of the observed data by modelling the paleoclimatic temperature process as the solution of an equation of the type $(*)$ with some noise term whose nature has to be determined. To take account of meta- or multi-stability, it is natural to assume that the climatic potential function has (at least) two wells, their stable minima corresponding to one cold (basic glacial) and at least one warm state (plus an intermediate one). In this setting characteristics of the transition mechanism between climate states may be reformulated in terms of the exit problem from local attractor neighborhoods for solutions of the stochastic differential equation. This approach was taken in [Dit99a, Dit99b]. To account for a reasonable choice of random noise perturbing the system, a spectral analysis of real ice-core data was performed in [Dit99b]. The obtained spectral decomposition exhibits a strong α-stable component with $\alpha \approx 1.75$. The paper [Dit99a] is concerned with an analysis of the exit times of $(*)$ with an α-stable noise η and in the limit of small ε, performed on a physical level of rigor, with the help of a fractional Fokker-Planck equation.

Climate dynamics is not the only source of stochastic models in which α-stable noise appears. For example, it was shown in [SMB97, BS02], that the thermally activated motion of the test particle along a polymer in three-dimensional space is subject to $\frac{1}{2}$-stable motion due to the polymer's self-intersections. In recent years, Lévy noise sources have been playing an increasingly important role in models of financial markets (see for example [ER99]).

In this paper we consider the equation $(*)$ driven by the Lévy process L which is the sum of a standard Brownian motion and an α-stable Lévy motion. Our approach of the asymptotic laws in the small noise limit of exit times from bounded intervals or intervals which are unbounded from one side is purely probabilistic and completely avoids fractional Fokker-Planck equations. We understand it as a first step towards a complete understanding of transition patterns of Lévy-driven dynamical systems in bi- or multi-stable potentials. The mathematical challenge consists in a large deviation analysis for exit times

replacing the classical theory of Freidlin-Wentzell [FW98] for diffusions with Brownian noise. We base it on a noise intensity dependent decomposition of L into a sum of two independent processes: a compound Poisson with large jumps on the one hand, and a sum of the Brownian motion and a Lévy motion with small jumps on the other hand. Given such a decomposition for small noise intensity ε, the main idea of our analysis is to prove that asymptotically exits from the considered domains are due to large jumps of the first component, while the second component is not able to perturb the deterministic trajectory of solutions of $(*)$ without noise essentially. For this reason, the usual picture of a particle that has to climb a potential well being pushed by a Brownian motion in order to exit a domain, which captures the system's behavior for Gaussian noise, changes drastically here. Instead of the height of the potential well, a large jump to exit just takes note of the distance from the domain's boundary. Pure horizontal distances replace geometric quantities related to the potential in the large deviations' estimates for exit times in the α-stable Lévy case. Also, the mean values of the exit times change essentially in comparison to the Gaussian noise setting: instead of Kramers' times we obtain exit times of the order of $\varepsilon^{-\alpha}$ in the small noise limit, i.e. times of *polynomial* instead of *exponential* dependence on ε. Our approach can be extended to more general heavy-tailed Lévy processes.

The structure of the paper is as follows. In section 1 we give a heuristic discussion of the asymptotic exit law based on the decomposition of the α-stable Lévy noise perturbing our system, and the heuristic picture that the small jump component does not essentially affect the asymptotic behavior. In section 2 we underpin this heuristic picture with mathematical rigor in proving that trajectories of the deterministic system and the one in which only the small noise component is admitted to perturb are asymptotically very close. This crucial observation is used in section 3 to derive in a rather technical way upper and lower estimates for the law of the exit time of a bounded interval. In sections 4 and 5 these results are transferred to the setting of intervals which are bounded on one side. This requires the possibility for the deterministic trajectory to return from $-\infty$ in finite time.

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Preliminaries and notation

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ we consider a stochastic differential equation driven by a Lévy noise of intensity ε:

$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\, ds + \varepsilon L_t, \quad \varepsilon > 0. \tag{1}$$

In general, a Lévy process is known to be a random process with independent and stationary increments, which is continuous in probability and possesses rcll paths. It is completely determined by its one-dimensional distributions which are infinitely divisible and characterized by the Lévy-Hinčin formula. In this paper we assume that

$$\mathbf{E}e^{i\lambda L_1} = \exp\left\{-d\frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y \mathbb{I}\{|y|<1\})\frac{dy}{|y|^{1+\alpha}}\right\}, \tag{2}$$

that is L is a sum of a standard Brownian motion with variance $d \geq 0$ and an independent α-stable Lévy motion with $0 < \alpha < 2$. More information on Lévy processes can be obtained from [Ber98, Sat99]. Since a Lévy process is a semimartingale, the standard theory of stochastic integration applies to equation (1), see [SG03, Pro04] for more details. Throughout this paper we assume that the underlying filtration fulfills the *usual conditions* in the sense of [Pro04], i.e. the filtration $(\mathcal{F}_t)_{t\geq 0}$ consists of σ-algebras which are complete with respect to $\mathbf{P}$ and is right-continuous.

The Lévy measure of L is given by $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$, $y \neq 0$. It is heavy-tailed and has infinite mass for all $\alpha \in (0, 2)$, due to a strong intensity of small jumps.

We impose some geometric conditions on the potential function U. First, we assume that U has a 'parabolic' shape with its non-degenerate global minimum at the origin, i.e. $U'(x)x \geq 0$, $U(0) = 0$, $U'(x) = 0$ iff $x = 0$, and $U''(0) = M > 0$. Further, to guarantee the existence of a strong unique solution of (1) on $\mathbb{R}$ we demand that U' is at least locally Lipschitz and increases faster than a linear function at $\pm\infty$ (see also [SG03, Pro04]). Moreover, in order to obtain some fine small-noise approximations of X^ε in Section 2, we need that $U \in \mathcal{C}^3$ in some sufficiently large interval containing the origin.

We shall study the first exit problem for the process X^ε from bounded and unbounded intervals in the small noise limit $\varepsilon \to 0$. In fact, we consider two cases.

(B) Let $I=[-b,a]$, $a,b>0$, and define the first exit time from I as

$$\sigma(\varepsilon)=\inf\{t\geq 0\,:\,X^\varepsilon_t\notin[-b,a]\}. \tag{3}$$

(U) Let $J=(-\infty,a]$, $a>0$, and assume that for some $c_1,c_2>0$ the regularity condition $U(x)=c_1|x|^{2+c_2}$, $x\to-\infty$, holds (see Remark 4.1 on page 21). In this case we study the one-sided counterpart of $\sigma(\varepsilon)$ defined by

$$\tau(\varepsilon)=\inf\{t\geq 0\,:\,X^\varepsilon_t>a\}. \tag{4}$$

We shall investigate the laws of $\sigma(\varepsilon)$ and $\tau(\varepsilon)$, in particular their mean values, as $\varepsilon\to 0$.

As a main tool of our analysis, we decompose the Lévy process L into ε-dependent small and large jump components. In mathematical terms, we represent the process L at any time t as a sum of two independent processes $L_t=\xi^\varepsilon_t+\eta^\varepsilon_t$, with characteristic functions

$$\begin{aligned}
\mathbf{E}e^{i\lambda\xi^\varepsilon_1}&=\exp\left\{-d\frac{\lambda^2}{2}+\int_{\mathbb{R}\setminus\{0\}}(e^{i\lambda y}-1-i\lambda y\mathbb{I}\{|y|<1\})\mathbb{I}\{|y|\leq\frac{1}{\sqrt{\varepsilon}}\}\frac{dy}{|y|^{1+\alpha}}\right\},\\
\mathbf{E}e^{i\lambda\eta^\varepsilon_1}&=\exp\left\{\int_{\mathbb{R}\setminus\{0\}}(e^{i\lambda y}-1)\mathbb{I}\{|y|>\frac{1}{\sqrt{\varepsilon}}\}\frac{dy}{|y|^{1+\alpha}}\right\}.
\end{aligned} \tag{5}$$

The Lévy measures corresponding of the processes ξ^ε and η^ε are

$$\nu^\varepsilon_\xi(\,\cdot\,)=\nu(\,\cdot\,\cap\{0<|y|\leq\frac{1}{\sqrt{\varepsilon}}\}),\quad \nu^\varepsilon_\eta(\,\cdot\,)=\nu(\,\cdot\,\cap\{|y|>\frac{1}{\sqrt{\varepsilon}}\}). \tag{6}$$

The process ξ^ε has an infinite Lévy measure with support $[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]\setminus\{0\}$, and makes infinitely many jumps on any time interval of positive length. The absolute value of its jumps does not exceed $1/\sqrt{\varepsilon}$. It will be explained later in Remark 3.1 on page 20 why the threshold $1/\sqrt{\varepsilon}$ is chosen.

The Lévy measure $\nu^\varepsilon_\eta(\cdot)$ of η^ε is finite. Denote

$$\beta_\varepsilon=\nu^\varepsilon_\eta(\mathbb{R})=\int_{\mathbb{R}\setminus[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]}\frac{dy}{|y|^{1+\alpha}}=\frac{2}{\alpha}\varepsilon^{\alpha/2}. \tag{7}$$

Then, η^ε is a compound Poisson process with intensity β_ε, and jumps distributed according to the law $\beta_\varepsilon^{-1}\nu^\varepsilon_\eta(\cdot)$.

Denote τ_k, $k\geq 0$, the arrival times of the jumps of η^ε with $\tau_0=0$. Let $T_k=\tau_k-\tau_{k-1}$ denote the inter-jump periods, and $W_k=\eta^\varepsilon_{\tau_k}-\eta^\varepsilon_{\tau_k-}$ the jump heights of η^ε. Then, the three processes $(T_k)_{k\geq1}$, $(W_k)_{k\geq1}$, and ξ^ε are independent. Moreover,

$$\begin{aligned}
&\mathbf{P}(T_k\geq u)=\int_u^\infty\beta_\varepsilon e^{-\beta_\varepsilon s}\,ds=e^{-\beta_\varepsilon u},\quad u\geq 0,\\
&\mathbf{E}T_k=\frac{1}{\beta_\varepsilon}=\frac{\alpha}{2}\varepsilon^{-\alpha/2}\to\infty\text{ as }\varepsilon\to 0,\\
&\mathbf{P}(W_k\in A)=\frac{1}{\beta_\varepsilon}\int_A\mathbb{I}\{|y|>\frac{1}{\sqrt{\varepsilon}}\}\frac{1}{|y|^{1+\alpha}}\,dy,\quad\text{for any Borel set}\quad A\subseteq\mathbb{R}.
\end{aligned} \tag{8}$$

Due to the strong Markov property, for any stopping time τ the process $\xi^\varepsilon_{t+\tau}-\xi^\varepsilon_\tau$, $t\geq 0$, is also a Lévy process with the same law as ξ^ε.

For $k\geq 1$ consider the processes

$$\begin{aligned}
\xi^k_t&=\xi^\varepsilon_{t+\tau_{k-1}}-\xi^\varepsilon_{\tau_{k-1}},\\
x^k_t(x)&=x-\int_0^t U'(x^k_{s-})\,ds+\varepsilon\xi^k_t,\quad t\in[0,T_k].
\end{aligned} \tag{9}$$

In our notation, for $x\in\mathbb{R}$,

$$\begin{aligned}
X^\varepsilon_t&=x^1_t(x)+\varepsilon W_1\mathbb{I}\{t=T_1\},\quad t\in[0,T_1],\\
X^\varepsilon_{t+\tau_1}&=x^2_t(x^1_{\tau_1}+\varepsilon W_1)+\varepsilon W_2\mathbb{I}\{t=T_2\},\quad t\in[0,T_2],\\
&\cdots\\
X^\varepsilon_{t+\tau_{k-1}}&=x^k_t(x^{k-1}_{\tau_{k-1}}+\varepsilon W_{k-1})+\varepsilon W_k\mathbb{I}\{t=T_k\},\quad t\in[0,T_k].
\end{aligned} \tag{10}$$

Finally, we denote by $Y(x)$ the deterministic function solving the non-perturbed version of (1)

$$Y_t(x)=x-\int_0^t U'(Y_s(x))\,ds,\quad x\in\mathbb{R}. \tag{11}$$

1 Heuristic derivation of the main result and comparison with Gaussian case

In this section we shall provide the skeleton of a heuristic derivation of our main result on the asymptotic law of the exit time from a bounded interval. In the subsequent two sections a rigorous underpinning of these arguments will be given.

On the interval $[0, T_k]$ and for $x \in \mathbb{R}$ let us consider $Y(x)$ and $x^k(x)$. These processes satisfy the equations

$$\begin{aligned} x_t^k(x) &= x - \int_0^t U'(x_{s-}^k(x))\, ds + \varepsilon \xi_t^k, \\ Y_t(x) &= x - \int_0^t U'(Y_s(x))\, ds, \quad t \in [0, T_k], \end{aligned} \tag{12}$$

while the law of T_k is an exponential with parameter β_ε.

The process ξ^k being the part of L with the small jumps, our analysis will be based on comparisons of the trajectories of x^k and Y. If they are close, e.g. if for some $\gamma > 0$, $\mathbf{P}_x(\sup_{0 \le s \le T_k} |x_s^k - Y_s(x)| > \varepsilon^\gamma)$ is small enough as $\varepsilon \to 0$, we can apply the following reasoning.

For any $x \in I$, the deterministic solution $Y(x)$ converges exponentially fast to the stable attractor 0. Define the relaxation time $R(x, \varepsilon)$ the process Y needs to reach an ε^γ-neighborhood of 0 from an arbitrary point $x \in I$. Then, as a separation of variables argument in (11) implies, for some $\mu_1 > 0$,

$$R(x, \varepsilon) \le \max\{\int_{-b}^{-\varepsilon^\gamma} \frac{dy}{-U'(y)}, \int_{\varepsilon^\gamma}^{a} \frac{dy}{U'(y)}\} \le \mu_1 |\ln \varepsilon|, \quad 0 < \varepsilon \le \varepsilon_0. \tag{13}$$

Since $Y(x)$ and $x^k(x)$ are close on the interjump interval $[0, T_k]$ for all $k \ge 0$ and all $x \in I$, X^ε can leave I only at one of the time instants τ_k while jumping by the distance εW_k.

If the process has not left I at jump number $k-1$, it waits for the next possibility to jump at the end of a random exponentially distributed time period T_k. Since T_k (on average) is essentially larger than the bound on the 'relaxation' time $\mu_1 |\ln \varepsilon|$, $\varepsilon \to 0$, this means that X^ε jumps from a small neighborhood of the attractor 0 (see Fig. 1) .

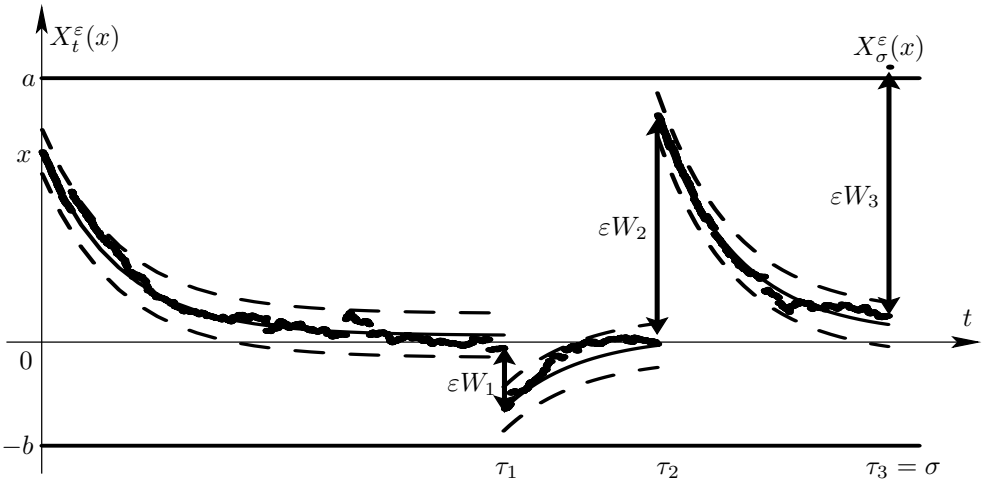

Fig. 1: A sample solution of the stochastic differential equation (1).

Therefore, the following heuristic estimate makes clear, that in the small noise limit $\sigma(\varepsilon)$ is an exponentially distributed random variable with parameter $\varepsilon^\alpha \theta/\alpha$, with

$$\theta = \frac{1}{a^\alpha} + \frac{1}{b^\alpha}. \tag{14}$$

Indeed, for all $k \ge 1$, $\tau_k = \sum_{j=1}^k T_j$ has a Gamma law with the density $\beta_\varepsilon e^{-\beta_\varepsilon t} \frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}$. Hence for $u \ge 0$

$$
\begin{aligned}
\mathbf{P}_x(\sigma(\varepsilon) > u) &\approx \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u) \cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k) \\
&= \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u) \cdot \mathbf{P}(\varepsilon W_1 \in I, \dots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \sum_{k=1}^{\infty} \int_u^{\infty} \beta_\varepsilon e^{-\beta_\varepsilon t} \frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}\, dt \cdot (1 - \mathbf{P}(\varepsilon W_1 \notin I))^{k-1} \cdot \mathbf{P}(\varepsilon W_1 \notin I) \\
&= \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I) \int_u^{\infty} e^{-\beta_\varepsilon t} \sum_{k=1}^{\infty} \frac{(\beta_\varepsilon t)^{k-1}(1 - \mathbf{P}(\varepsilon W_1 \notin I))^{k-1}}{(k-1)!}\, dt \\
&= \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I) \int_u^{\infty} e^{-\beta_\varepsilon t} e^{\beta_\varepsilon t (1 - \mathbf{P}(\varepsilon W_1 \notin I))}\, dt = e^{-u\beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I)} = e^{-u\varepsilon^\alpha \theta/\alpha}
\end{aligned}
\tag{15}
$$

For deriving the last formula we use the equations

$$
\begin{aligned}
\mathbf{P}(\varepsilon W_1 \notin I) = \mathbf{P}(W_1 < -\frac{b}{\varepsilon} \text{ or } W_1 > \frac{a}{\varepsilon}) &= \frac{1}{\beta_\varepsilon} \left(\int_{\frac{a}{\varepsilon}}^{\infty} \frac{dy}{y^{1+\alpha}} + \int_{\frac{b}{\varepsilon}}^{\infty} \frac{dy}{y^{1+\alpha}} \right) \\
&= \frac{1}{\beta_\varepsilon} \frac{1}{\alpha} \left[\left(\frac{\varepsilon}{a}\right)^\alpha + \left(\frac{\varepsilon}{b}\right)^\alpha \right] = \frac{\varepsilon^\alpha}{\beta_\varepsilon \alpha} \theta.
\end{aligned}
\tag{16}
$$

The mean value of $\sigma(\varepsilon)$ may be obtained immediately from (15), or independently by the following reasoning:

$$
\begin{aligned}
\mathbf{E}_x \sigma(\varepsilon) &\approx \sum_{k=1}^{\infty} \mathbf{E}\tau_k \cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k) = \sum_{k=1}^{\infty} k\mathbf{E}T_k \cdot \mathbf{P}(\varepsilon W_1 \in I, \dots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I) \sum_{k=1}^{\infty} k(1 - \mathbf{P}(\varepsilon W_1 \notin I))^{k-1} = \frac{\beta_\varepsilon}{\mathbf{P}(\varepsilon W_1 \notin I)} = \frac{\alpha}{\varepsilon^\alpha} \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1}.
\end{aligned}
\tag{17}
$$

The aim of this paper is to make these heuristic arguments rigorous. This is done by proving the following Theorems.

Theorem 1.1 *There exist positive constants ε_0, γ, δ, and $C > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the following asymptotics holds*

$$
\exp\left\{ -u\varepsilon^\alpha \frac{\theta}{\alpha}(1 + C\varepsilon^\delta) \right\} (1 - C\varepsilon^\delta) \le \mathbf{P}_x(\sigma(\varepsilon) > u) \le \exp\left\{ -u\varepsilon^\alpha \frac{\theta}{\alpha}(1 - C\varepsilon^\delta) \right\} (1 + C\varepsilon^\delta) \tag{18}
$$

uniformly for all $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$ and $u \ge 0$, where $\theta = \frac{1}{a^\alpha} + \frac{1}{b^\alpha}$. Consequently,

$$
\mathbf{E}_x \sigma(\varepsilon) = \frac{\alpha}{\varepsilon^\alpha} \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1} (1 + \mathcal{O}(\varepsilon^\delta)) \tag{19}
$$

uniformly for all $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$.

Theorem 1.2 *There exist positive constants ε_0, γ, δ, and $C > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the following asymptotics holds*

$$
\exp\left\{ -u\frac{\varepsilon^\alpha}{\alpha a^\alpha}(1 + C\varepsilon^\delta) \right\} (1 - C\varepsilon^\delta) \le \mathbf{P}_x(\tau(\varepsilon) > u) \le \exp\left\{ -u\frac{\varepsilon^\alpha}{\alpha a^\alpha}(1 - C\varepsilon^\delta) \right\} (1 + C\varepsilon^\delta) \tag{20}
$$

uniformly for all $x \le a - \varepsilon^\gamma$ and $u \ge 0$. Consequently,

$$
\mathbf{E}_x \tau(\varepsilon) = \alpha \frac{a^\alpha}{\varepsilon^\alpha}(1 + \mathcal{O}(\varepsilon^\delta)) \tag{21}
$$

uniformly for all $x \le a - \varepsilon^\gamma$.

It is interesting to compare the results stated above with their well-known counterparts for diffusions driven by the Brownian motion of small intensity ε. Together with (1) consider the diffusion $\hat{X}^\varepsilon$ which solves the stochastic differential equation

$$
\hat{X}_t^\varepsilon = x - \int_0^t U'(\hat{X}_s^\varepsilon)\, ds + \varepsilon W_t, \tag{22}
$$

where W is a standard one-dimensional Brownian motion, and U is the same potential as in (1). For the diffusion $\hat{X}^\varepsilon$ we define the first exit time of the interval I by

$$\hat{\sigma}(\varepsilon) = \inf\{t \geq 0 \,:\, \hat{X}^\varepsilon_t \notin [-b, a]\}. \tag{23}$$

Then the following statements hold for $\hat{\sigma}(\varepsilon)$ in the limit of small ε.

1. The first exit time $\hat{\sigma}(\varepsilon)$ is exponentially large in ε^{-2}. Assume for definiteness, that $U(a) < U(-b)$. Then for any $\delta > 0, x \in I$, according to [FW98]:

$$\mathbf{P}_x(e^{(2U(a)-\delta)/\varepsilon^2} < \hat{\sigma}(\varepsilon) < e^{(2U(a)+\delta)/\varepsilon^2}) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{24}$$

Moreover, $\varepsilon^2 \ln \mathbf{E}_x \hat{\sigma}(\varepsilon) \to 2U(a)$.

The mean of the first exit time can be calculated more explicitly (Kramers' law) [Kra40, Sch80, BEGK02]:

$$\mathbf{E}_x \hat{\sigma}(\varepsilon) \approx \frac{\varepsilon\sqrt{\pi}}{U'(a)\sqrt{U''(0)}} e^{2U(a)/\varepsilon^2} \tag{25}$$

2. The normalized first exit time is exponentially distributed [Wil82, Day83, BGK02]: for $u \geq 0$

$$\mathbf{P}_x\left(\frac{\hat{\sigma}(\varepsilon)}{\mathbf{E}_x\hat{\sigma}(\varepsilon)} > u\right) \to e^{-u} \quad \text{as} \quad \varepsilon \to 0, \tag{26}$$

uniformly in x on compact subsets of $(-b, a)$.

As we see, $\hat{\sigma}(\varepsilon)$ and $\sigma(\varepsilon)$ have different orders of growth as $\varepsilon \to 0$. The exit times of the α-stable driven processes are much shorter because of the presence of large jumps which occur with polynomially small probability. To leave the interval, the diffusion $\hat{X}^\varepsilon$ has to overcome a potential barrier of height either $U(-b)$ or $U(a)$. So in the case considered here, $\hat{X}^\varepsilon_{\hat{\sigma}(\varepsilon)} = a$ with an overwhelming probability. The diffusion 'climbs' up in the potential landscape. This also explains why the pre-factor in (25) depends on geometric properties of U such as the slope at the exit point and the curvature at the local minimum, the place where the diffusion spends most of its time before exit.

The process X^ε on the contrary uses the possibility to exit the interval at one large jump. This is the reason why the asymptotic exit time depends mainly on the distance between the stable point 0 and the interval's boundaries. The potential's geometry does not play a big role for the low order approximations of the exit time $\sigma(\varepsilon)$. Although it is important for the proof, it does not appear in the pre-factors of the mean exit time in (19) and remains hidden in the error terms.

In the purely Gaussian case, to obtain the law of $\hat{\sigma}(\varepsilon)$, the theory of partial differential equations is used. In fact, the probability $p_\varepsilon(x, u) = \mathbf{P}_x(\hat{\sigma}(\varepsilon) > u)$ as a function of x and u satisfies a backward Kolmogorov equation (parabolic partial differential equation) with appropriate boundary conditions. The function $p_\varepsilon(x, u)$ can be (at least in $\mathbb{R}$) expanded in a Fourier series with respect to the eigensystem of the diffusion's infinitesimal generator. Then, one concludes that $p_\varepsilon(x, u) \approx e^{-\lambda_1^\varepsilon u}$, where λ_1^ε is the first eigenvalue. Further one shows, that $\lambda_1^\varepsilon \mathbf{E}_x\hat{\sigma}(\varepsilon) \to 1$ as $\varepsilon \to 0$. In contrast to this, in the present paper we obtain results without any use of operator theory. This suggests that some asymptotic spectral properties of the integro-differential operators corresponding to the process X^ε can be formulated from the probabilistic estimates obtained here. This is the subject of future research.

Perturbations of the deterministic dynamical systems by small infinitely divisible noises were considered e.g. in [FW98], however in a different setting. There, the small parameter ε was responsible for the *simultaneous* scaling of jump size and jump intensity. As the simplest example of such a perturbation one can consider a compensated Poisson process with jump size of the order ε and jump intensity of the order $1/\varepsilon$ (see also [Bor67]). In such a case the dynamics in the limit corresponds to the one of the system perturbed by white noise, i.e. the probabilities of the rare (exit) events are exponentially small in ε and the characteristic time scales are exponentially large. Note that the perturbations considered in the present paper are of quite a different nature. We only scale the jump sizes, while the jump intensities stay unchanged.

2 Deviations from the deterministic trajectory: exit from bounded interval

In this section we estimate the deviation of the solutions of the stochastic differential equation driven by the small-jump process $\varepsilon\xi^\varepsilon$ from the deterministic trajectory on random time intervals of exponentially

distributed length. We show that the probabilities for at least polynomially small deviations are polynomially small in ε in the small noise limit. This rigorously underpins the starting point of our heuristic derivation of the exit law.

For $x \in [-b, a]$ consider solutions x^ε and Y of the equations

$$\begin{aligned} x_t^\varepsilon(x) &= x - \int_0^t U'(x_{s-}^\varepsilon(x))\, ds + \varepsilon \xi_t^\varepsilon, \\ Y_t(x) &= x - \int_0^t U'(Y_s(x))\, ds. \end{aligned} \tag{27}$$

The goal of this section is to prove the following estimate.

Proposition 2.1 *Let T_ε be exponentially distributed with parameter β_ε, and independent of ξ^ε. Let $c > 0$, $\gamma = \frac{2-\alpha}{5}$. Then there is $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ and $x \in [-b, a]$ the inequality*

$$\mathbf{P}_x(\sup_{t \in [0, T_\varepsilon]} |x_t^\varepsilon - Y_t(x)| \ge c\varepsilon^\gamma) \le C\varepsilon^{(\alpha+\gamma)/2} \tag{28}$$

holds.

In order to prove the Proposition, we shall make use of the following Lemma in which the estimation of the deviation from the deterministic trajectory is executed on a bounded deterministic time interval.

Lemma 2.1 *Let $T \ge 0$, $c > 0$ and $\gamma = \frac{2-\alpha}{5}$ ($0 < \gamma < \frac{2}{5}$). Then there exist positive numbers ε_0 and C such that for all $0 < \varepsilon \le \varepsilon_0$ and $x \in [-b, a]$ the inequality*

$$\mathbf{P}_x(\sup_{[0,T]} |x_t^\varepsilon - Y_t(x)| \ge c\varepsilon^\gamma) \le CT\varepsilon^{\alpha+\gamma/2} \tag{29}$$

holds.

Proof of Proposition 2.1: We apply Lemma 2.1 and the definition of β_ε to obtain for all $x \in [-b, a]$ and $\varepsilon \le \varepsilon_0$

$$\begin{aligned} \mathbf{P}_x(\sup_{[0,T_\varepsilon]} |x_t^\varepsilon - Y_t(x)| \ge c\varepsilon^\gamma) &= \int_0^\infty \mathbf{P}_x(\sup_{[0,\tau]} |x_t^\varepsilon - Y_t(x)| \ge c\varepsilon^\gamma) \beta_\varepsilon e^{-\beta_\varepsilon \tau}\, d\tau \\ &\le C'\varepsilon^{\alpha+\gamma/2} \int_0^\infty \tau \beta_\varepsilon e^{-\beta_\varepsilon \tau}\, d\tau \le C\varepsilon^{(\alpha+\gamma)/2}. \end{aligned} \tag{30}$$

■

The proof of Lemma 2.1 is performed in three Lemmas in the sequel. It extensively uses the following geometric properties of the potential U:

1. The deterministic trajectories $Y_t(x)$, $x \in [-b, a]$ converge to zero as $t \to \infty$ due to the property $U'(x)x > 0$ for $x \ne 0$.
2. The curvature of the potential at $x = 0$ is positive. In small neighborhoods of zero we have $U(x) = \frac{M}{2}x^2 + o(x^2)$. Consequently Y decays there like e^{-Mt}, and the dynamics of x^ε reminds of the dynamics of a process of Ornstein-Uhlenbeck type.

We now prepare our rigorous analysis by an asymptotic expansion of the random trajectories of x^ε around the deterministic one of Y. To this end, fix some $\delta > 0$ small enough which will be specified later and define

$$\hat{T} = \max\{\int_{-b}^{-\delta} \frac{dy}{-U'(y)}, \int_\delta^a \frac{dy}{U'(y)}\}. \tag{31}$$

$\hat{T}$ has the property that for all $x \in [-a, b]$ and $t \ge \hat{T}$, $|Y_t(x)| \le \delta$, i.e. after $\hat{T}$ the trajectory of $Y(x)$ is within a δ-neighborhood of the origin. We next consider the representation of the process x^ε in powers of ε of the form

$$x^\varepsilon(x) = Y(x) + \varepsilon Z^\varepsilon(x) + R^\varepsilon(x), \tag{32}$$

where Z^ε is the first approximation of x^ε in powers of ε satisfying the stochastic differential equation

$$Z_t^\varepsilon(x) = -\int_0^t U''(Y_s(x)) Z_{s-}^\varepsilon(x)\, ds + \xi_t^\varepsilon. \tag{33}$$

The solution to this equation is explicitly given by

$$Z_t^\varepsilon(x) = \int_0^t e^{-\int_s^t U''(Y_u(x))\, du}\, d\xi_s^\varepsilon. \tag{34}$$

Integration by parts results in the following representation for Z^ε:

$$Z_t^\varepsilon(x) = \xi_t^\varepsilon - \int_0^t \xi_{s-}^\varepsilon U''(Y_s(x)) e^{-\int_s^t U''(Y_u(x))\,du}\,ds. \tag{35}$$

For $x = 0$, $Y_t(x) = 0$ for all $t \geq 0$, and $Z^\varepsilon(0)$ is a process of the Ornstein-Uhlenbeck type starting at zero and given by the equation

$$Z_t^\varepsilon(0) = \xi_t^\varepsilon - M\int_0^t \xi_{s-}^\varepsilon e^{-M(t-s)}\,ds. \tag{36}$$

Lemma 2.2 *There is a universal constant $C_Z > 2$ such that for any $T > 0$, $x \in [-b, a]$ and $\varepsilon > 0$*

$$\sup_{s\in[0,T]} |Z_s^\varepsilon(x)| \leq C_Z \sup_{s\in[0,T]} |\xi_s^\varepsilon|, \quad \mathbf{P}_x\text{-a.s.}. \tag{37}$$

Proof: Obviously, for $t \leq T$

$$|Z_t^\varepsilon(x)| \leq \sup_{t\in[0,T]} |\xi_t^\varepsilon| \left(1 + \sup_{t\in[0,T]} \int_0^t |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds\right). \tag{38}$$

We shall show that the integral in the parenthesis is uniformly bounded. Fix some $\delta > 0$ small enough such that for some $0 < m_1 < m_2$ the inequality $m_1 < \inf_{|x|<\delta} U''(x) \leq \sup_{|x|<\delta} U''(x) < m_2$ holds. This implies, that $m_1 < U''(Y_t(x)) < m_2$ for all $x \in I$, and $t \geq \hat{T}$. Let

$$C_1 = \max_{x\in I} \int_0^{\hat{T}} |U''(Y_s(x))| e^{-\int_s^{\hat{T}} U''(Y_u(x))\,du}\,ds. \tag{39}$$

Consider an arbitrary $t \geq \hat{T}$. Then

$$\begin{aligned} &\int_0^t |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds \\ &\qquad = \int_0^{\hat{T}} |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds + \int_{\hat{T}}^t |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds. \end{aligned} \tag{40}$$

Let us estimate the first summand in (40). We have

$$\begin{aligned} \int_0^{\hat{T}} |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds &= e^{-\int_{\hat{T}}^t U''(Y_u(x))\,du} \int_0^{\hat{T}} |U''(Y_s(x))| e^{-\int_s^{\hat{T}} U''(Y_u(x))\,du}\,ds \\ &\leq e^{-m_1(t-\hat{T})} C_1 \leq C_1. \end{aligned} \tag{41}$$

The second summand in (40) is estimated analogously:

$$\int_{\hat{T}}^t |U''(Y_s(x))| e^{-\int_s^t U''(Y_u(x))\,du}\,ds \leq m_2 \int_{\hat{T}}^t e^{-m_1(t-s)}\,ds \leq \frac{m_2}{m_1}. \tag{42}$$

Taking $C_Z = \max\{2, C_1 + \frac{m_2}{m_1}\}$ completes the proof.

■

To estimate the remainder term R^ε we need finer smoothness properties of the potential U. However, the following Lemma shows that this restriction only has to hold locally.

Lemma 2.3 *There exists $C > 0$ such that for all $x \in [-b, a]$ and $T > 0$,*

$$\sup_{t\in[0,T]} |R_t^\varepsilon(x)| \leq C \tag{43}$$

a.s. on the event $\{\omega : \sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon(\omega)| < 1\}$.

Proof: By hypothesis we know that for any $t \geq 0, x \in [-b, a]$ we have $-b \leq Y_t(x) \leq a$. Moreover, on $\{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| < 1\}$ we have $\sup_{t\in[0,T]} |\varepsilon Z_t^\varepsilon| < C_Z$ due to Lemma 2.2. Recall that U' increases at least linearly at infinity. This guarantees the existence of $C > 0$ such that for any $y \in [-b, a], z \in [-C_Z, C_Z]$ we have

$$-U'(y + z + C) + U'(y) + U''(y)z < 0.$$

Hence for any $T > 0, T \geq \tau > 0, x \in [-b, a]$ the inequality

$$-U'(Y_\tau(x) + \varepsilon Z_{\tau-}^\varepsilon(x) + C) + U'(Y_\tau(x)) + U''(Y_\tau(x))\varepsilon Z_{\tau-}^\varepsilon(x) < 0. \tag{44}$$

holds on the event $\{\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|<1\}$. Now assume there is some $x\in[-b,a]$, and some (smallest) $\tau\in[0,T]$ such that $R^\varepsilon_\tau(x)=C$. Observe that the rest term satisfies the integral equation

$$R^\varepsilon_t(x)=\int_0^t f(R^\varepsilon_s(x),Y_s(x),\varepsilon Z^\varepsilon_{s-}(x))\,ds \tag{45}$$

with the smooth integrand

$$f(R,Y,\varepsilon Z)=-U'(Y+\varepsilon Z+R)+U'(Y)+U''(Y)(\varepsilon Z).$$

This implicitly says that R^ε is an absolutely continuous function of time. By definition of τ, we have

$$0\le D^+R^\varepsilon_t(x)|_{t=\tau}=-U'(Y_\tau(x)+\varepsilon Z^\varepsilon_{\tau-}(x)+C)+U'(Y_\tau(x))+U''(Y_\tau(x))\varepsilon Z^\varepsilon_{\tau-}(x)<0,$$

a contradiction. A similar reasoning applies under the assumption $R^\varepsilon_\tau(x)=-C$. This completes the proof. ■

Lemma 2.3 has a very convenient consequence. It states precisely that the solution process x^ε, with initial state confined to $[-b,a]$, stays bounded by a deterministic constant on sets of the form $\{\omega:\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t(\omega)|<1\}$. Therefore, in the small noise limit, only local properties of U are relevant to our analysis.

We next obtain a finer estimate of the remainder term R^ε on the time interval $[0,\hat T]$.

Lemma 2.4 *There exits $C_{\hat T}>0$ such that for $0<T\le\hat T$*

$$\sup_{s\in[0,T]}|R^\varepsilon_s(x)|\le C_{\hat T}(\sup_{s\in[0,T]}|\varepsilon\xi^\varepsilon_s|)^2,\quad \mathbf{P}_x\text{-a.s.},\tag{46}$$

on the event $\{\omega:\sup_{[0,T]}|\varepsilon\xi^\varepsilon_t|<1\}$ uniformly for $x\in[-b,a]$ and $\varepsilon>0$.

Proof: Using Lemma 2.3, choose $K>0$ such that on the event $\{\omega:\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|<1\}$ the processes $x^\varepsilon(x),\varepsilon Z^\varepsilon(x),R^\varepsilon(x)$ take their values in $[-K,K]$ as long as time runs in $[0,T]$. For $t\le T\le\hat T$, the rest term R^ε satisfies the following integral equation:

$$\begin{aligned}R^\varepsilon_t(x)&=\int_0^t\left[-U'(Y_s(x)+\varepsilon Z^\varepsilon_{s-}(x)+R^\varepsilon_s(x))+U'(Y_s(x))+U''(Y_s(x))\varepsilon Z^\varepsilon_{s-}(x)\right]ds\\&=-\int_0^t\left[U'(Y_s(x)+\varepsilon Z^\varepsilon_{s-}(x)+R^\varepsilon_s(x))-U'(Y_s(x)+\varepsilon Z^\varepsilon_{s-}(x))\right]ds\\&\quad-\int_0^t\left[U'(Y_s(x)+\varepsilon Z^\varepsilon_{s-}(x))-U'(Y_s(x))-U''(Y_s(x))\varepsilon Z^\varepsilon_{s-}(x)\right]ds\\&=-\int_0^t U''(\theta^1_{s-})R^\varepsilon_s(x)\,ds-\int_0^t\tfrac12U^{(3)}(\theta^2_{s-})(\varepsilon Z^\varepsilon_{s-}(x))^2\,ds\end{aligned}\tag{47}$$

with appropriate θ^1_s, $\theta^2_s\in[-K,K]$. Thus

$$|R^\varepsilon_t(x)|\le\int_0^t L|R^\varepsilon_s(x)|\,ds+\tfrac12TLC_Z^2(\sup_{[0,T]}|\varepsilon\xi^\varepsilon_t|)^2.\tag{48}$$

An application of Gronwall's lemma yields the final estimates

$$\begin{aligned}&|R^\varepsilon_t(x)|\le\tfrac12TLC_Z^2e^{TL}(\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|)^2\le\tfrac12\hat TLC_Z^2e^{\hat TL}(\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|)^2=C_{\hat T}(\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|)^2,\\&\sup_{t\in[0,T]}|R^\varepsilon_t(x)|\le C_{\hat T}(\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|)^2.\end{aligned}\tag{49}$$

■

Now we derive a uniform estimate of the rest term R^ε on time intervals longer than $\hat T$.

Lemma 2.5 *There exist positive constants C_R and $C_E\le1$ such that for $T\ge0$*

$$\sup_{s\in[0,T]}|R^\varepsilon_s(x)|\le C_R(\sup_{s\in[0,T]}|\varepsilon\xi^\varepsilon_s|)^2\tag{50}$$

on the event

$$E_T=\{\omega:\sup_{t\in[0,T]}|\varepsilon\xi^\varepsilon_t|<C_E\}\tag{51}$$

uniformly for $x\in[-b,a]$.

Proof: Fix some positive $T \geq \hat{T}$ and let $\omega \in \{\omega : \sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| < 1\}$. Again, using Lemma 2.3, choose $K > 0$ such that on the event $\{\omega : \sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| < 1\}$ the processes $x^\varepsilon(x), \varepsilon Z^\varepsilon(x), R^\varepsilon(x)$ take their values in $[-K, K]$ as long as time runs in $[0, T]$. The rest term R^ε satisfies the integral equation

$$R_t^\varepsilon(x) = R_{\hat{T}}^\varepsilon(x) + \int_{\hat{T}}^t f(R_s^\varepsilon(x), Y_s(x), \varepsilon Z_{s-}^\varepsilon(x))\, ds \tag{52}$$

with

$$f(R, Y, \varepsilon Z) = -U'(Y + \varepsilon Z + R) + U'(Y) + U''(Y)(\varepsilon Z).$$

Moreover, R^ε is an absolutely continuous function of time. We write the Taylor expansion for the integrand f with some $\theta \in [-K, K]$:

$$\begin{aligned} f(R, Y, \varepsilon Z) &= -U'(Y + \varepsilon Z + R) + U'(Y) + U''(Y)(\varepsilon Z) \\ &= -U'(Y) - U''(Y)(R + \varepsilon Z) - \frac{U^{(3)}(\theta)}{2}(R + \varepsilon Z)^2 + U'(Y) + U''(Y)(\varepsilon Z) \\ &= -U''(Y)R - \frac{U^{(3)}(\theta)}{2}(R + \varepsilon Z)^2 \end{aligned} \tag{53}$$

Since $U \in \mathcal{C}^3$, $|U^{(3)}|$ is bounded, say by L, on $[-K, K]$. Using the inequality $(R + \varepsilon Z)^2 \leq 2(R^2 + \varepsilon^2 Z^2)$ we obtain that for all R, Y and Z such that $L|\varepsilon Z| < A$

$$\begin{aligned} f(R, Y, \varepsilon Z) &\leq -U''(Y)R + LR^2 + L(\varepsilon Z)^2 < -U''(Y)R + LR^2 + A = g^+(R, Y) \\ f(R, Y, \varepsilon Z) &\geq -U''(Y)R - LR^2 - L(\varepsilon Z)^2 > -U''(Y)R - LR^2 - A = g^-(R, Y). \end{aligned} \tag{54}$$

With the help of lemma 2.2 this immediately implies for $\hat{T} < t \leq T$ that

$$\begin{aligned} R_t^\varepsilon(x) &= R_{\hat{T}}^\varepsilon(x) + \int_{\hat{T}}^t f(R_s^\varepsilon(x), Y_s(x), \varepsilon Z_{s-}^\varepsilon(x))\, ds < R_{\hat{T}}^\varepsilon(x) + \int_{\hat{T}}^t g^+(R_s^\varepsilon(x), Y_s(x))\, ds, \\ R_t^\varepsilon(x) &= R_{\hat{T}}^\varepsilon(x) + \int_{\hat{T}}^t f(R_s^\varepsilon(x), Y_s(x), \varepsilon Z_{s-}^\varepsilon(x))\, ds > R_{\hat{T}}^\varepsilon(x) + \int_{\hat{T}}^t g^-(R_s^\varepsilon(x), Y_s(x))\, ds. \end{aligned} \tag{55}$$

with $A = D\varepsilon^2(\sup_{t\in[0,T]} |\xi_t^\varepsilon|)^2$ and a constant $D > 2LC_Z^2$ which will be specified later.

To estimate the rest term R^ε on $[\hat{T}, T]$ we apply the subsequent comparison Lemma 2.6. Consider an event

$$E_1 = \{\omega : |R_{\hat{T}}^\varepsilon(x)| < \frac{A}{m_2}\} \supseteq \{\omega : C_{\hat{T}}(\sup_{t\in[0,\hat{T}]} |\varepsilon\xi_t^\varepsilon|)^2 < \frac{D}{m_2}(\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon|)^2\} \tag{56}$$

Thus, setting $D > \max\{2LC_Z^2, C_{\hat{T}} m_2\}$ we obtain that $\mathbf{P}_x(E_1) = 1$, $x \in [-b, a]$ and the conditions of Lemma 2.6 are fulfilled on the event

$$E_2 = \{\omega : m_1^2 - 4AL > 0\} = \{\omega : \sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| < \frac{m_1}{2\sqrt{LD}}\}. \tag{57}$$

On E_2 the following inequality holds a.s. for $x \in [-b, a]$:

$$\sup_{t\in[\hat{T},T]} |R_t^\varepsilon(x)| \leq \frac{2D}{m_2}(\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon|)^2. \tag{58}$$

Thus denoting $C_R = \max\{C_{\hat{T}}, \frac{m_1}{2\sqrt{LD}}\}$ and $C_E = \min\{\frac{2D}{m_2}, 1\}$ we may finish the proof.

■

Lemma 2.6 (Comparison lemma) *Let $T \geq 0$, Y be a smooth function on $[0, T]$ and Z a rcll function on $[0, T]$. Consider the integral equation*

$$R_t = R_0 + \int_0^t f(R_s, Y_s, Z_{s-})\, ds, \quad t \in [0, T], \tag{59}$$

with a smooth function f satisfying

$$\begin{aligned} &g^-(R, Y) < f(R, Y, Z) < g^+(R, Y), \quad R, Y, Z \in \mathbb{R}, \\ &g^\pm(R, Y) = -U''(Y)R \pm LR^2 \pm A, \quad R, Y \in \mathbb{R}, \quad \textit{with} \quad L, A > 0. \end{aligned} \tag{60}$$

Moreover, let $0 < m_1 < U''(Y_t) < m_2$, $t \in [0, T]$, and $m_1^2 - 4AL > 0$. Then for $0 < t \leq T$ the following holds:

1. *if* $R_0 < \frac{A}{m_2}$ *then* $R_t < \frac{2A}{m_2}$;
2. *if* $R_0 > -\frac{A}{m_1}$ *then* $R_t > -\frac{2A}{m_1}$.

This yields, that if $|R_0| < \frac{A}{m_2}$ *then* $\sup_{t\in[0,T]} |R_t| < \frac{2A}{m_2}$.

Proof: To prove the first statement, together with (59) consider the Riccati equation

$$r_t^+ = R_0 + \int_0^t g^+(r_s^+, Y_s)\, ds, \quad t \in [0,T]. \tag{61}$$

Under the conditions of the lemma, it is enough to prove two statements:

a) $R_t < r_t^+$ for $0 < t \le T$.

b) $r_t^+ < \frac{2A}{m_1}$ for $t \ge 0$.

To show **a)** we note that at the starting point $t = 0$,

$$D^+R_t\Big|_{t=0} = \lim_{h\downarrow 0} \frac{R_h - R_0}{h} = f(R_0, Y_0, Z_0) < g^+(R_0, Y_0) = \dot r_t^+\Big|_{t=0}, \tag{62}$$

consequently it follows from the continuity of R and r^+ that $r_t^+ > R_t$ for at least positive and small t. Assume there exists $\tau = \inf\{t > 0 : R_\tau = r_\tau^+\}$ such that $\tau \in (0, T]$. At the point τ the left derivative of R is necessarily not less than the derivative of r^+ which leads to the following contradiction:

$$\begin{aligned} D^-R_t\Big|_{t=\tau} &= \lim_{h\downarrow 0} \frac{R_\tau - R_{\tau-h}}{h} = f(R_\tau, Y_\tau, Z_{\tau-}) \ge \dot r_t^+\Big|_{t=\tau} = g^+(r_\tau^+, Y_\tau), \\ f(R_\tau, Y_\tau, Z_{\tau-}) &= f(r_\tau^+, Y_\tau, Z_{\tau-}) < g^+(r_\tau, Y_\tau). \end{aligned} \tag{63}$$

To prove **b)**, we compare r^+ with the stationary solution of the autonomous Riccati equation

$$p_t = p_0 + \int_0^t (-m_2 p_s + L p_s^2 + A)\, ds, \quad t \ge 0. \tag{64}$$

Equation (64) has two positive stationary solutions at which the integrand vanishes:

$$p_0 = p^\pm = \frac{m_2}{2L}\left(1 \pm \sqrt{1 - \frac{4AL}{m_2^2}}\right). \tag{65}$$

Repeating the comparison argument used for **a)**, we obtain that if $R_0 < p^-$, then $r_t^+ < p^-$, $t \in [0,T]$. Applying the elementary inequalities

$$\frac{x}{2} \le 1 - \sqrt{1-x} \le x, \quad x \in [0,1], \tag{66}$$

to p^- we obtain that $\frac{A}{m_2} \le p^- \le \frac{2A}{m_2} < \frac{2A}{m_1}$. This guarantees that for $R_0 < \frac{A}{m_2}$, the solution of (59) does not exceed $\frac{2A}{m_1}$.

The proof of the second statement is analogous. ■

Proof of Lemma 2.1: Let $T \ge 0$ and $x \in [-b, a]$. Choose E_T according to Lemma 2.5. Then there exists ε_0 such that for $\varepsilon \le \varepsilon_0$ the following holds:

$$\begin{aligned} &\{\sup_{t\in[0,T]} |x_t^\varepsilon(x) - Y_t(x)| \ge c\varepsilon^\gamma\} = \{\sup_{t\in[0,T]} |\varepsilon Z_t^\varepsilon(x) + R_t^\varepsilon(x)| \ge c\varepsilon^\gamma\} \\ &\subseteq \{\sup_{t\in[0,T]} |\varepsilon Z_t^\varepsilon(x)| \ge \tfrac{c\varepsilon^\gamma}{2}\} \cup \{\sup_{t\in[0,T]} |R_t^\varepsilon(x)| \ge \tfrac{c\varepsilon^\gamma}{2}\} \\ &\subseteq \{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| \ge \tfrac{c\varepsilon^\gamma}{2C_Z}\} \cup \left[\{\sup_{t\in[0,T]} |R_t^\varepsilon(x)| \ge \tfrac{c\varepsilon^\gamma}{2}\} \cap E_T\right] \cup \left[\{\sup_{t\in[0,T]} |R_t^\varepsilon(x)| \ge \tfrac{c\varepsilon^\gamma}{2}\} \cap E_T^c\right] \\ &\subseteq \{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| \ge \tfrac{c\varepsilon^\gamma}{2C_Z}\} \cup \{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| \ge \tfrac{c\varepsilon^{\gamma/2}}{2\sqrt{C_R}}\} \cup \{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| \ge C_E\} \\ &= \{\sup_{t\in[0,T]} |\varepsilon\xi_t^\varepsilon| \ge \tfrac{c\varepsilon^\gamma}{2C_Z}\}. \end{aligned} \tag{67}$$

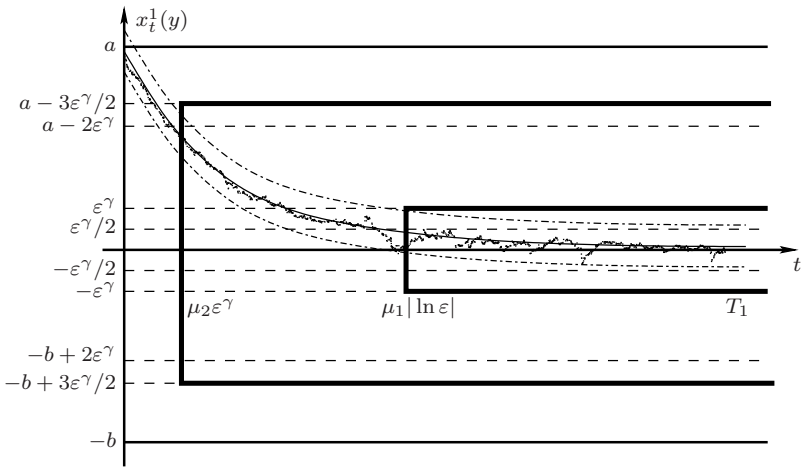

Fig. 2: The dynamics of $x^1(y)$ under condition $\sup|x_t^1(y)-Y_t(y)|\le\frac{\varepsilon^\gamma}{2}$.

Consequently, with the help of Kolmogorov's inequality we obtain for small ε and some $C>0$

$$\begin{aligned}\mathbf{P}_x(\sup_{t\in[0,T]}|x_t^\varepsilon-Y_t(x)|\ge c\varepsilon^\gamma)&\le\mathbf{P}(\sup_{t\in[0,T]}|\varepsilon\xi_t^\varepsilon|\ge\frac{c}{2C_Z}\varepsilon^\gamma)\le\frac{4C_Z^2}{c^2\varepsilon^{2\gamma}}\mathbf{E}(\varepsilon\xi_T^\varepsilon)^2\\&=T\frac{4C_Z^2}{c^2}(\frac{2}{2-\alpha}\varepsilon^{1+\alpha/2-2\gamma}+d\varepsilon^{2-2\gamma})\le CT\varepsilon^{1+\alpha/2-2\gamma}.\end{aligned}\tag{68}$$

This completes the proof. ■

3 The law of $\sigma(\varepsilon)$

For the purposes of this rather technical section we introduce the following notation. Denote $W_0=T_0=0$, $x^1(0)=x$, and write $\mathbb{I}A$ for the indicator function of a measurable set A. Let $I_q^\pm$ denote the exterior and interior neighborhoods of I defined by $I_q^\pm=[-b\mp q,a\pm q]$ for $0\le q<\min\{a,b\}$.

Throughout this section we use the following constants. Chose μ_1, μ_2 and $\varepsilon_0>0$ such that for all $y\in I$ the following holds for $\varepsilon\in(0,\varepsilon_0]$, (see Fig. 2):
1. $|Y_t(y)|\le\frac{\varepsilon^\gamma}{2}$ for $t\ge\mu_1|\ln\varepsilon|$. Consequently, denoting $z_t^k(y)=x_t^k(y)-Y_t(y)$, we have $|x_t^k(y)|\le\varepsilon^\gamma$ for $t\ge\mu_1|\ln\varepsilon|$ under the condition $|z_t^k(y)|\le\frac{\varepsilon^\gamma}{2}$.
2. $Y_t(y)\in I_{2\varepsilon^\gamma}^-$ for all $t\ge\mu_2\varepsilon^\gamma$. Consequently, $|x_t^k(y)|\in I_{3\varepsilon^\gamma/2}^-$ for $t\ge\mu_2\varepsilon^\gamma$ under the condition $|z_t^k(y)|\le\frac{\varepsilon^\gamma}{2}$. Let us show that μ_2 with this property exists. Indeed, for all $y\in I$, $|Y_t(y)|$ is strictly decreasing ($y\ne0$), and $Y_t(0)\equiv0$. Moreover, for $-b\le y_1<y_2\le a$, $Y_t(y_1)<Y_t(y_2)$. Let $r(x,\varepsilon)=\inf\{t\ge0:Y_t(x)\in I_{2\varepsilon^\gamma}^-\}$. Then by comparison for all $x\in I$

$$r(x,\varepsilon)\le\max\{\int_{a-2\varepsilon^\gamma}^a\frac{dy}{U'(y)},\int_{-b}^{-b+2\varepsilon^\gamma}\frac{dy}{|U'(y)|}\}\le\mu_2\varepsilon^\gamma,\quad 0<\varepsilon\le\varepsilon_0.\tag{69}$$

3.1 Upper estimate

In this subsection we give estimates of $\mathbf{P}_x(\sigma(\varepsilon)>u)$ from above as $\varepsilon\to0$, $u>0$. They are comprised in the following Proposition with a rather technical proof. Recall that γ has been chosen according to α in Proposition 2.1.

Proposition 3.1 *Let $\delta=\min\{\alpha/2,\gamma/2\}$. There exist constants $\varepsilon_0>0$ and $C>0$ such that for all $0<\varepsilon\le\varepsilon_0$, $x\in[-b+\varepsilon^\gamma,a-\varepsilon^\gamma]$ and $u\ge0$ the following inequality holds*

$$\mathbf{P}(\sigma(\varepsilon)>u)\le\exp\left\{-u\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right](1-C\varepsilon^\delta)\right\}(1+C\varepsilon^\delta).\tag{70}$$

Proof: For $x \in I$, we use the following obvious inequality

$$\begin{aligned}\mathbf{P}_x(\sigma(\varepsilon) > u) &= \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u)\mathbf{P}_x(\sigma(\varepsilon) = \tau_k) + \mathbf{P}_x(\sigma(\varepsilon) > u | \sigma(\varepsilon) \in (\tau_{k-1}, \tau_k))\mathbf{P}_x(\sigma(\varepsilon) \in (\tau_{k-1}, \tau_k)) \\ &\leq \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u)\Big[\mathbf{P}_x(\sigma(\varepsilon) = \tau_k) + \mathbf{P}_x(\sigma(\varepsilon) \in (\tau_{k-1}, \tau_k))\Big].\end{aligned} \tag{71}$$

Then for any $x \in I$ and $k \in \mathbb{N}$, applying the independence and law properties of the processes x^j, $j \in \mathbb{N}$, the following chain of inequalities is deduced which results in a factorization formula for the probability under estimation (compare with (15)):

$$\begin{aligned}\mathbf{P}_x(\sigma(\varepsilon) = \tau_k) &= \mathbf{E}_x \mathbb{I}\{X^\varepsilon_s \in I, s \in [0, \tau_k), X^\varepsilon_{\tau_k} \notin I\} \\ &= \mathbf{E}_x \prod_{j=1}^{k-1} \mathbb{I}\{x^j_s(x^{j-1}_{T_{j-1}} + \varepsilon W_{j-1}) \in I, s \in [0, T_j], x^j_{T_j}(x^{j-1}_{T_{j-1}} + \varepsilon W_{j-1}) + \varepsilon W_j \in I\} \\ &\qquad \times \mathbb{I}\{x^k_s(x^{k-1}_{T_{k-1}} + \varepsilon W_{k-1}) \in I, s \in [0, T_k], x^k_{T_k}(x^{k-1}_{T_{k-1}} + \varepsilon W_{k-1}) + \varepsilon W_k \notin I\} \\ &\leq \mathbf{E} \prod_{j=1}^{k-1} \sup_{y \in I} \mathbb{I}\{x^j_s(y) \in I, s \in [0, T_j], x^j_{T_j}(y) + \varepsilon W_j \in I\} \\ &\qquad \times \sup_{y \in I} \mathbb{I}\{x^k_s(y) \in I, s \in [0, T_k], x^k_{T_k}(y) + \varepsilon W_k \notin I\} \\ &= \prod_{j=1}^{k-1} \mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^j_s(y) \in I, s \in [0, T_j], x^j_{T_j}(y) + \varepsilon W_j \in I\}\right] \\ &\qquad \times \mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^k_s(y) \in I, s \in [0, T_k], x^k_{T_k}(y) + \varepsilon W_k \notin I\}\right] \\ &= \left(\mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^1_s(y) \in I, s \in [0, T_1], x^1_{T_1}(y) + \varepsilon W_1 \in I\}\right]\right)^{k-1} \\ &\qquad \times \mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^1_s(y) \in I, s \in [0, T_1], x^1_{T_1}(y) + \varepsilon W_1 \notin I\}\right].\end{aligned} \tag{72}$$

Analogously we estimate the probability to exit between the k-th arrival times of the compound Poisson process η^ε, $k \in \mathbb{N}$. Here we distinguish two cases.

In the first case $k = 1, x \in I^-_{\varepsilon^\gamma}$. Then

$$\begin{aligned}&\mathbf{P}_x(\sigma(\varepsilon) \in (\tau_0, \tau_1)) = \mathbf{P}_x(\sigma(\varepsilon) \in (0, T_1)) = \mathbf{E}_x \mathbb{I}\{X^\varepsilon_s \notin I \text{ for some } s \in (0, T_1)\} \\ &\leq \mathbf{E}\left[\sup_{y \in I^-_{\varepsilon^\gamma}} \mathbb{I}\{x^1_s(y) \notin I \text{ for some } s \in [0, T_1]\}\right],\end{aligned}$$

In the second case $k \geq 2, x \in I$. Then

$$\begin{aligned}&\mathbf{P}_x(\sigma(\varepsilon) \in (\tau_{k-1}, \tau_k)) = \mathbf{E}_x \mathbb{I}\{X^\varepsilon_s \in I, s \in [0, \tau_{k-1}], X^\varepsilon_s \notin I \text{ for some } s \in (\tau_{k-1}, \tau_k)\} \\ &= \mathbf{E}_x \prod_{j=1}^{k-1} \mathbb{I}\{x^j_s(x^{j-1}_{T_{j-1}} + \varepsilon W_{j-1}) \in I, s \in [0, T_j], x^j_{T_j}(x^{j-1}_{T_{j-1}} + \varepsilon W_{j-1}) + \varepsilon W_j \in I\} \\ &\qquad \times \mathbb{I}\{x^k_s(x^{k-1}_{T_{k-1}} + \varepsilon W_{k-1}) \notin I \text{ for some } s \in [0, T_k)\} \\ &\leq \mathbf{E} \prod_{j=1}^{k-2} \sup_{y \in I} \mathbb{I}\{x^j_s(y) \in I, s \in (0, T_j], x^j_{T_j}(y) + \varepsilon W_j \in I\} \times \\ &\times \sup_{y \in I} \mathbb{I}\{x^{k-1}_s(y) \in I, s \in (0, T_{k-1}], x^{k-1}_{T_{k-1}}(y) + \varepsilon W_{k-1} \in I, x^k_s(x^{k-1}_{T_{k-1}}(y) + \varepsilon W_{k-1}) \notin I \text{ for some } s \in [0, T_k]\} \\ &= \left(\mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^1_s(y) \in I, s \in [0, T_1], x^1_{T_1}(y) + \varepsilon W_1 \in I\}\right]\right)^{k-2} \\ &\times \mathbf{E}\left[\sup_{y \in I} \mathbb{I}\{x^1_s(y) \in I, s \in (0, T_1], x^1_{T_1}(y) + \varepsilon W_1 \in I, x^2_s(x^1_{T_1}(y) + \varepsilon W_1) \notin I \text{ for some } s \in [0, T_2]\}\right].\end{aligned} \tag{73}$$

Next we specify separately in four steps the further estimation for the four different events appearing in the formulae for $\mathbf{P}_x(\sigma(\varepsilon)=\tau_k)$ and $\mathbf{P}_x(\sigma(\varepsilon)\in(\tau_{k-1},\tau_k))$.

Step 1. Consider $\mathbb{I}\{x_s^1(y)\in I, s\in[0,T_1], x_{T_1}^1(y)+\varepsilon W_1\in I\}$. For $y\in I$, we may estimate

$$
\begin{aligned}
&\mathbb{I}\{x_s^1(y)\in I, s\in[0,T_1], x_{T_1}^1(y)+\varepsilon W_1\in I\}\\
&=\mathbb{I}\{x_s^1(y)\in I, s\in[0,T_1], x_{T_1}^1(y)+\varepsilon W_1\in I\}\left(\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}\}\right)\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, x_{T_1}^1(y)+\varepsilon W_1\in I\}\\
&=\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, x_{T_1}^1(y)+\varepsilon W_1\in I\}\left(\mathbb{I}\{|\varepsilon W_1|\le\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}\}\right)\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{|\varepsilon W_1|\le\tfrac{\varepsilon^\gamma}{2}\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, x_{T_1}^1(y)+\varepsilon W_1\in I, T_1\ge\mu_1|\ln\varepsilon|\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, x_{T_1}^1(y)+\varepsilon W_1\in I, T_1<\mu_1|\ln\varepsilon|\}\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{|\varepsilon W_1|\le\tfrac{\varepsilon^\gamma}{2}\}\\
&+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, \varepsilon W_1\in I_{\varepsilon^\gamma}^+\}+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|\}\\
&=\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{\varepsilon W_1\in I_{\varepsilon^\gamma}^+\}+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|\}.
\end{aligned}
\tag{74}
$$

Step 2. Consider $\mathbb{I}\{x_s^1(y)\in I, s\in[0,T_1], x_{T_1}^1(y)+\varepsilon W_1\notin I\}$. For $y\in I$, we may estimate

$$
\begin{aligned}
&\mathbb{I}\{x_s^1(y)\in I, s\in[0,T_1], x_{T_1}^1(y)+\varepsilon W_1\notin I\}\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, x_{T_1}^1(y)+\varepsilon W_1\notin I\}\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|\le\tfrac{\varepsilon^\gamma}{2}, T_1\ge\mu_2\varepsilon^\gamma, x_{T_1}^1(y)+\varepsilon W_1\notin I\}\ (=0)\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|\le\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_2\varepsilon^\gamma, x_{T_1}^1(y)+\varepsilon W_1\notin I\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\ge\mu_1|\ln\varepsilon|, x_{T_1}^1(y)+\varepsilon W_1\notin I\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|, x_{T_1}^1(y)+\varepsilon W_1\notin I\}\\
&\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\mathbb{I}\{T_1<\mu_2\varepsilon^\gamma\}+\mathbb{I}\{\varepsilon W_1\notin I_{\varepsilon^\gamma}^-\}+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|\}.
\end{aligned}
\tag{75}
$$

Step 3. Consider $\mathbb{I}\{x_s^1(y)\notin I \text{ for some } s\in[0,T_1]\}$. For $y\in I_{\varepsilon^\gamma}^-$, we may estimate

$$
\begin{aligned}
&\mathbb{I}\{x_s^1(y)\notin I \text{ for some } s\in[0,T_1]\}\le\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|>\tfrac{\varepsilon^\gamma}{2}\}+\\
&\mathbb{I}\{\sup_{s\in[0,T_1]}|z_s^1(y)|\le\tfrac{\varepsilon^\gamma}{2}, x_s^1(y)\notin I \text{ for some } s\in[0,T_1]\}(=0)
\end{aligned}
\tag{76}
$$

Step 4. Consider $\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}$. For $y\in I$, we may estimate

$$\begin{aligned}
&\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}\\
&=\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}\\
&+\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I\backslash I^-_{\varepsilon^\gamma}, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}\\
&\le \mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\cdot \sup_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^2_s(y)\notin I \text{ for some } s\in[0,T_2]\}\\
&+\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I\backslash I^-_{\varepsilon^\gamma}\}\\
&\le \sup_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^2_s(y)\notin I \text{ for some } s\in[0,T_2]\}+\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I\backslash I^-_{\varepsilon^\gamma}\}.
\end{aligned}\tag{77}$$

The first term in the resulting expression in the Step 4 is identical to the expression handled in Step 3, while the second term requires an inessential modification of the estimation in Step 1.

Now we apply (74), (75), (76) and (77) to estimate the expectations treated in Steps 1, 2, 3 and 4 above. In what follows, $c.$ and $C.$ denote appropriate positive constants. Fix also $0<\delta<\min\{\gamma/2,\alpha(1-\gamma),\alpha/2\}=\min\{\gamma/2,\alpha/2\}$.

Step 1. Estimate $\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}+\varepsilon W_1\in I\}\right]$. We get

$$\begin{aligned}
&\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}+\varepsilon W_1\in I\}\right]\\
&\le \mathbf{P}(\sup_{[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2})+\mathbf{P}(\varepsilon W_1\in I^+_{\varepsilon^\gamma})+\mathbf{P}(|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2})\mathbf{P}(T_1<\mu_1|\ln\varepsilon|)\\
&\le c_1\varepsilon^{(\alpha+\gamma)/2}+1-\frac{\varepsilon^{\alpha/2}}{2}\left[\frac{1}{(a+\varepsilon^\gamma)^\alpha}+\frac{1}{(b+\varepsilon^\gamma)^\alpha}\right]+c_2\varepsilon^{\alpha(3/2-\gamma)}|\ln\varepsilon|\\
&\le 1-\varepsilon^{\alpha/2}\frac{\theta}{2}(1-C_1\varepsilon^\delta).
\end{aligned}\tag{78}$$

Step 2. Estimate $\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}+\varepsilon W_1\notin I\}\right]$. In fact,

$$\begin{aligned}
&\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}+\varepsilon W_1\notin I\}\right]\\
&\le \mathbf{P}(\sup_{[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2})+\mathbf{P}(T_1<\mu_2\varepsilon^\gamma)\\
&+\mathbf{P}(\varepsilon W_1\notin I^-_{\varepsilon^\gamma})+\mathbf{P}(|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2})\mathbf{P}(T_1<\mu_1|\ln\varepsilon|)\\
&\le c_1\varepsilon^{(\alpha+\gamma)/2}+c_5\varepsilon^{\alpha/2+\gamma}+\frac{\varepsilon^{\alpha/2}}{2}\left[\frac{1}{(a-\varepsilon^\gamma)^\alpha}+\frac{1}{(b-\varepsilon^\gamma)^\alpha}\right]+c_2\varepsilon^{\alpha(3/2-\gamma)}|\ln\varepsilon|\\
&\le \varepsilon^{\alpha/2}\frac{\theta}{2}(1+C_2\varepsilon^\delta)
\end{aligned}\tag{79}$$

Step 3. Estimate $\mathbf{E}\left[\sup_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^1_s(y)\notin I \text{ for some } s\in[0,T_1]\}\right]$. We have

$$\mathbf{E}\left[\sup_{y\in I-_{\varepsilon^\gamma}}\mathbb{I}\{x^1_s(y)\notin I \text{ for some } s\in[0,T_1]\}\right]\le \mathbf{P}(\sup_{[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2})\le c_1\varepsilon^{(\alpha+\gamma)/2}\le \varepsilon^{\alpha/2}\cdot C_3\varepsilon^\delta \tag{80}$$

Step 4. Estimate
$\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}\right]$. We finally obtain

$$\begin{aligned}
&\mathbf{E}\left[\sup_{y\in I}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I, x^2_s(x^1_{T_1}(y)+\varepsilon W_1)\notin I \text{ for some } s\in[0,T_2]\}\right]\\
&\le \mathbf{P}(\sup_{[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2})\\
&+\mathbf{P}(\sup_{[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2})+\mathbf{P}(\varepsilon W_1\in[-b-\varepsilon^\gamma,-b+2\varepsilon^\gamma])+\mathbf{P}(\varepsilon W_1\in[a-2\varepsilon^\gamma,a+\varepsilon^\gamma])\\
&+\mathbf{P}(|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2})\mathbf{P}(T_1<\mu_1|\ln\varepsilon|)\le \varepsilon^{\alpha/2}\cdot C_4\varepsilon^\delta.
\end{aligned}\tag{81}$$

Then for $x \in I^-_{\varepsilon^\gamma}$, $0 < \varepsilon \le \varepsilon_0$, and some positive C_5,

$$\begin{aligned}
\mathbf{P}_x(\sigma(\varepsilon) > u) &\le \mathbf{P}(\tau_1 > u)\left[\varepsilon^{\alpha/2}\frac{\theta}{2}(1 + C_2\varepsilon^\delta) + \varepsilon^{\alpha/2}\cdot C_3\varepsilon^\delta\right] \\
&+ \sum_{k=2}^{\infty}\mathbf{P}(\tau_k > u)\left(1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_1\varepsilon^\delta)\right)^{k-1}\varepsilon^{\alpha/2}\frac{\theta}{2}\left[\frac{\frac{2}{\theta}C_4\varepsilon^\delta}{1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_1\varepsilon^\delta)} + 1 + C_2\varepsilon^\delta\right] \\
&\le \sum_{k=1}^{\infty}\int_u^\infty \beta_\varepsilon e^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}\,dt\left(1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_5\varepsilon^\delta)\right)^{k-1}\varepsilon^{\alpha/2}\frac{\theta}{2}(1 + C_5\varepsilon^\delta) \\
&\le \varepsilon^{\alpha/2}\frac{\theta}{2}(1 + C_5\varepsilon^\delta)\int_u^\infty \beta_\varepsilon e^{-\beta_\varepsilon t\varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_5\varepsilon^\delta)}\,dt \\
&\le \frac{1 + C_5\varepsilon^\delta}{1 - C_5\varepsilon^\delta}\exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 - C_5\varepsilon^\delta)\right\} \le \exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 - C_5\varepsilon^\delta)\right\}(1 + C_6\varepsilon^\delta) \\
&\le \exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 - C\varepsilon^\delta)\right\}(1 + C\varepsilon^\delta)
\end{aligned} \tag{82}$$

with $C = \max\{C_5, C_6\}$.

In the previous formula we have changed summation and integration. This can be done due to the uniform convergence of the series $\sum_{k=1}^\infty \beta_\varepsilon e^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}\left[1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_5\varepsilon^\delta)\right]^{k-1}$ for all $t \ge 0$ and $\varepsilon \le \varepsilon_0$. Indeed, let t_k^* be the coordinate of the maximum of the density of the (β_ε, k)-Gamma distribution. For $k \ge 2$, it is easy to see that $t_k^* = \frac{k-1}{\beta_\varepsilon}$ Then, with help of Stirling's formula we obtain, that

$$0 \le \beta_\varepsilon e^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!} \le \beta_\varepsilon e^{-(k-1)}\frac{(k-1)^{k-1}}{(k-1)!} \sim \frac{1}{\sqrt{2\pi}}\frac{\beta_\varepsilon}{\sqrt{k-1}}, \quad k \to \infty. \tag{83}$$

Then, for all $\varepsilon \le \varepsilon_0$,

$$\sum_{k=1}^{\infty}\beta_\varepsilon e^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}\left[1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_5\varepsilon^\delta)\right]^{k-1} \le c_1\frac{\beta_\varepsilon}{\varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_5\varepsilon^\delta)} \le c, \tag{84}$$

where the constant c does not depend on t and ε. Thus uniform convergence follows from dominated convergence.

■

3.2 Lower estimate

In this subsection we estimate $\mathbf{P}_x(\sigma(\varepsilon) > u)$ from below as $\varepsilon \to 0$, $u > 0$. This leads to the following Proposition with a rather technical proof again.

Proposition 3.2 *There exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$, $0 < \delta < \min\{\alpha/2, \gamma/2\}$, $x \in [-b + \varepsilon^\gamma, a - \varepsilon^\gamma]$ and $u \ge 0$ the estimate*

$$\mathbf{P}(\sigma(\varepsilon) > u) \ge \exp\left\{-u\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right](1 + C\varepsilon^\delta)\right\}(1 - C\varepsilon^\delta) \tag{85}$$

is valid.

Proof: We use the following inequality:

$$\begin{aligned}
\mathbf{P}_x(\sigma(\varepsilon) > u) &= \sum_{k=1}^{\infty}\mathbf{P}(\tau_k > u)\mathbf{P}_x(\sigma(\varepsilon) = \tau_k) + \mathbf{P}_x(\sigma(\varepsilon) > u | \sigma \in (\tau_{k-1}, \tau_k))\mathbf{P}_x(\sigma \in (\tau_{k-1}, \tau_k)) \\
&\ge \sum_{k=1}^{\infty}\mathbf{P}(\tau_k > u)\mathbf{P}_x(\sigma(\varepsilon) = \tau_k).
\end{aligned} \tag{86}$$

With arguments analogous to (72) we obtain the factorization

$$
\begin{aligned}
&\mathbf{P}_x(\sigma=\tau_k)=\mathbf{E}_x\mathbb{I}\{X^\varepsilon_s\in I, s\in[0,\tau_k), X^\varepsilon_{\tau_k}\notin I\}\\
&=\mathbf{E}_x\prod_{j=1}^{k-1}\mathbb{I}\{x^j_s(x^{j-1}_{T_{j-1}}+\varepsilon W_{j-1})\in I, s\in[0,T_j], x^j_{T_j}(x^{j-1}_{T_{j-1}}+\varepsilon W_{j-1})+\varepsilon W_j\in I\}\\
&\qquad\qquad\times\mathbb{I}\{x^k_s(x^{k-1}_{T_{k-1}}+\varepsilon W_{k-1})\in I, s\in[0,T_k], x^k_{T_k}(x^{k-1}_{T_{k-1}}+\varepsilon W_{k-1})+\varepsilon W_k\notin I\}\\
&\geq\mathbf{E}_x\prod_{j=1}^{k-1}\mathbb{I}\{x^j_s(x^{j-1}_{T_{j-1}}+\varepsilon W_{j-1})\in I, s\in[0,T_j], x^j_{T_j}(x^{j-1}_{T_{j-1}}+\varepsilon W_{j-1})+\varepsilon W_j\in I^-_{\varepsilon^\gamma}\}\\
&\qquad\qquad\times\mathbb{I}\{x^k_s(x^{k-1}_{T_{k-1}}+\varepsilon W_{k-1})\in I, s\in[0,T_k], x^k_{T_k}(x^{k-1}_{T_{k-1}}+\varepsilon W_{k-1})+\varepsilon W_k\notin I\}\\
&\geq\mathbf{E}\prod_{j=1}^{k-1}\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^j_s(y)\in I, s\in(0,T_j], x^j_{T_j}(y)+\varepsilon W_j\in I^-_{\varepsilon^\gamma}\}\\
&\qquad\qquad\times\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^k_s(y)\in I, s\in(0,T_k], x^k_{T_k}(y)+\varepsilon W_k\notin I\}\\
&=\prod_{j=1}^{k-1}\mathbf{E}\left[\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^j_s(y)\in I, s\in(0,T_j], x^j_{T_j}(y)+\varepsilon W_j\in I^-_{\varepsilon^\gamma}\}\right]\\
&\qquad\qquad\times\mathbf{E}\left[\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^k_s(y)\in I, s\in(0,T_k], x^k_{T_k}(y)+\varepsilon W_k\notin I\}\right]\\
&=\left(\mathbf{E}\left[\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\right]\right)^{k-1}\\
&\qquad\qquad\times\mathbf{E}\left[\inf_{y\in I^-_{\varepsilon^\gamma}}\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\notin I\}\right].
\end{aligned}
\tag{87}
$$

For $y\in I^-_{\varepsilon^\gamma}$, we next specify separately in two steps the further estimation for the two different events appearing in the formulae for $\mathbf{P}_x(\sigma(\varepsilon)=\tau_k)$.

Step 1. Consider the event $\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}$. We may estimate

$$
\begin{aligned}
&\mathbb{I}\{x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\\
&\geq\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|\leq\tfrac{\varepsilon^\gamma}{2}, x^1_s(y)\in I, s\in(0,T_1], x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\\
&\geq\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|\leq\tfrac{\varepsilon^\gamma}{2}, x^1_s(y)\in I, s\in(0,T_1], |\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_2\varepsilon^\gamma, x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|\leq\tfrac{\varepsilon^\gamma}{2}, x^1_s(y)\in I, s\in(0,T_1], |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_1|\ln\varepsilon|, x^1_{T_1}(y)+\varepsilon W_1\in I^-_{\varepsilon^\gamma}\}\\
&\geq\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|\leq\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_2\varepsilon^\gamma\}\\
&+\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|\leq\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_1|\ln\varepsilon|, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}\\
&=\mathbb{I}\{|\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_2\varepsilon^\gamma\}\\
&-\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_2\varepsilon^\gamma\}\\
&+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_1|\ln\varepsilon|, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}\\
&-\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_1|\ln\varepsilon|, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}\\
&=\mathbb{I}\{|\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}\}-\mathbb{I}\{|\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_2\varepsilon^\gamma\}-\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|\leq\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_2\varepsilon^\gamma\}\\
&+\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}-\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}\\
&-\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2}, |\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1\geq\mu_1|\ln\varepsilon|, \varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}\\
&\geq\mathbb{I}\{\varepsilon W_1\in I^-_{2\varepsilon^\gamma}\}-\mathbb{I}\{T_1<\mu_2\varepsilon^\gamma\}-2\mathbb{I}\{\sup_{s\in[0,T_1]}|z^1_s(y)|>\tfrac{\varepsilon^\gamma}{2}\}-\mathbb{I}\{|\varepsilon W_1|>\tfrac{\varepsilon^\gamma}{2}, T_1<\mu_1|\ln\varepsilon|\}.
\end{aligned}
\tag{88}
$$

Step 2. The event $\mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^k + \varepsilon W_1 \notin I\}$ may be estimated as follows:

$$\begin{aligned}
&\mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^k + \varepsilon W_1 \notin I\} \\
&\geq \mathbb{I}\{\sup_{s\in[0,T_1]} |z_s^1(y)| \leq \tfrac{\varepsilon^\gamma}{2}, x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^k + \varepsilon W_1 \notin I\} \\
&\geq \mathbb{I}\{\sup_{s\in[0,T_1]} |z_s^1(y)| \leq \tfrac{\varepsilon^\gamma}{2}, x_s^1(y) \in I, s \in (0,T_1], T_1 \geq \mu_1|\ln\varepsilon|, x_{T_1}^k + \varepsilon W_1 \notin I\} \\
&\geq \mathbb{I}\{\sup_{s\in[0,T_1]} |z_s^1(y)| \leq \tfrac{\varepsilon^\gamma}{2}, T_1 \geq \mu_1|\ln\varepsilon|, \varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\} \\
&= \mathbb{I}\{T_1 \geq \mu_1|\ln\varepsilon|, \varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\} - \mathbb{I}\{\sup_{s\in[0,T_1]} |z_s^1(y)| > \tfrac{\varepsilon^\gamma}{2}, T_1 \geq \mu_1|\ln\varepsilon|, \varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\} \\
&\geq \mathbb{I}\{\varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\} - \mathbb{I}\{T_1 < \mu_1|\ln\varepsilon|, \varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\} - \mathbb{I}\{\sup_{s\in[0,T_1]} |z_s^1(y)| > \tfrac{\varepsilon^\gamma}{2}, \varepsilon W_1 \notin I_{\varepsilon^\gamma}^+\}.
\end{aligned} \tag{89}$$

Now we apply (88) and (89) to estimate the expectations appearing in the formula for $\mathbf{P}_x(\sigma = \tau_k)$. In what follows, c_i and C_i denote appropriate positive constants. Fix also $0 < \delta < \min\{\gamma/2, \alpha(1-\gamma), \alpha/2\} = \min\{\gamma/2, \alpha/2\}$.

Step 1. Here we estimate $\mathbf{E}\left[\inf_{y\in I_{\varepsilon^\gamma}^-} \mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^1 + \varepsilon W_1 \in I_{\varepsilon^\gamma}^-\}\right]$. In fact, we obtain from employing results from sections 2 and 1

$$\begin{aligned}
&\mathbf{E}\left[\inf_{y\in I_{\varepsilon^\gamma}^-} \mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^1 + \varepsilon W_1 \in I_{\varepsilon^\gamma}^-\}\right] \\
&\geq \mathbf{P}(\varepsilon W_1 \in I_{2\varepsilon^\gamma}^-) - \mathbf{P}(T_1 < \mu_2\varepsilon^\gamma) - 2\mathbf{P}(\sup_{[0,T_1]} |z_s^1(y)| > \tfrac{\varepsilon^\gamma}{2}) - \mathbf{P}(|\varepsilon W_1| > \tfrac{\varepsilon^\gamma}{2}, T_1 < \mu_1|\ln\varepsilon|) \\
&\geq 1 - \frac{\varepsilon^{\alpha/2}}{2}\left[\frac{1}{(a-2\varepsilon^\gamma)^\alpha} + \frac{1}{(b-2\varepsilon^\gamma)^\alpha}\right] - c_1\varepsilon^{\alpha/2+\gamma} - 2c_2\varepsilon^{(\alpha+\gamma)/2} - c_3\varepsilon^{\alpha(3/2-\gamma)}|\ln\varepsilon| \\
&\geq 1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 + C_1\varepsilon^\delta).
\end{aligned} \tag{90}$$

Step 2. We next estimate $\mathbf{E}\left[\inf_{y\in I_{\varepsilon^\gamma}^-} \mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^k + \varepsilon W_1 \notin I\}\right]$, for which we obtain similarly

$$\begin{aligned}
&\mathbf{E}\left[\inf_{y\in I_{\varepsilon^\gamma}^-} \mathbb{I}\{x_s^1(y) \in I, s \in (0,T_1], x_{T_1}^k + \varepsilon W_1 \notin I\}\right] \geq \\
&\mathbf{P}(\varepsilon W_1 \notin I_{\varepsilon^\gamma}^+)\left(1 - \mathbf{P}(T_1 < \mu_1|\ln\varepsilon|) - \mathbf{P}(\sup_{[0,T_1]} |z_s^1(y)| > \tfrac{\varepsilon^\gamma}{2})\right) \\
&\geq \frac{\varepsilon^{\alpha/2}}{2}\left[\frac{1}{(a+\varepsilon^\gamma)^\alpha} + \frac{1}{(b+\varepsilon^\gamma)^\alpha}\right]\left(1 - c_3\varepsilon^{\alpha/2}|\ln\varepsilon| - c_2\varepsilon^{(\alpha+\gamma)/2}\right) \\
&\geq \varepsilon^{\alpha/2}\frac{\theta}{2}\left(1 - C_2\varepsilon^\delta\right).
\end{aligned} \tag{91}$$

Consequently, with $C_3 = \max\{C_1, C_2\}$, for $x \in I_{\varepsilon^\gamma}^-$, $0 < \varepsilon \leq \varepsilon_0$

$$\begin{aligned}
\mathbf{P}_x(\sigma(\varepsilon) > u) &\geq \sum_{k=1}^\infty \int_u^\infty \beta e^{-\beta t}\frac{(\beta t)^{k-1}}{(k-1)!}\,dt\left[1 - \varepsilon^{\alpha/2}\frac{\theta}{2}(1 + C_3\varepsilon^\delta)\right]^{k-1}\varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_3\varepsilon^\delta) \\
&= \varepsilon^{\alpha/2}\frac{\theta}{2}(1 - C_4\varepsilon^\delta)\int_u^\infty \beta e^{-\beta t\varepsilon^{\alpha/2}\frac{\theta}{2}(1+C_3\varepsilon^\delta)}\,dt \\
&\geq \frac{1 - C_4\varepsilon^\delta}{1 + C_3\varepsilon^\delta}\exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 + C_3\varepsilon^\delta)\right\} \geq \exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 + C_3\varepsilon^\delta)\right\}(1 - C_5\varepsilon^\delta) \\
&\geq \exp\left\{-u\varepsilon^\alpha\frac{\theta}{\alpha}(1 + C\varepsilon^\delta)\right\}(1 - C\varepsilon^\delta)
\end{aligned} \tag{92}$$

with $C = \max\{C_3, C_5\}$. See the end of the proof of Proposition 3.1 for the justification of switching the order of summation and integration in the above argument.

■

Proof of Theorem 1.1: The statement of Theorem 1.1 follows directly from Propositions 3.1 and 3.2.

■

Remark 3.1 The threshold $1/\sqrt{\varepsilon}$ for separating the two parts of L is not the only possible choice. Indeed, let us consider thresholds of the type $1/\varepsilon^\rho$ for $\rho > 0$. Then $\beta_\varepsilon = \int_{\mathbb{R}} \mathbb{I}\{|y| > \frac{1}{\varepsilon^\rho}\}\nu(dy) = \frac{2}{\alpha}\varepsilon^{\alpha\rho}$. ρ must satisfy some further conditions. Firstly, we demand that $\rho < 1$ so that we can easily calculate the probability

$$P(\varepsilon W_1 \notin [-b, a]) = \frac{\alpha}{2\varepsilon^{\alpha\rho}} \int_{\mathbb{R}\setminus[-\frac{b}{\varepsilon}, \frac{a}{\varepsilon}]} \mathbb{I}\{|y| > \frac{1}{\varepsilon^\rho}\} \frac{dy}{|y|^{1+\alpha}} = \mathcal{O}(\varepsilon^{\alpha(1-\rho)}). \tag{93}$$

This is the *characteristic* probability of our analysis. The probabilities of other relevant events should have smaller order in the small noise limit. This for example applies to the event that ξ^ε leaves the ε^γ-tube around the deterministic trajectory (see Proposition 2.1), and the event that $T_1 < \mu_2\varepsilon^\gamma$) (e.g. see (78), (79) and (80)). This leads to the following inequalities on ρ and γ: $0 < \rho < 1$, $\gamma > 0$, $\alpha(1-\rho) < 2 - 2\rho - 2\gamma$ and $\alpha(1-\rho) < \alpha\rho + \gamma$. Applying some algebra we rewrite these inequalities in the form

$$\begin{cases} \gamma < \frac{2-\alpha}{2}(1-\rho), & 0 < \rho < 1, \\ \gamma > \alpha(1-2\rho), & \gamma > 0. \end{cases} \tag{94}$$

The solution set (ρ, γ) is non-empty for all $\alpha \in (0, 2)$ and depends on α. However $\rho = \frac{1}{2}$ is the minimal value independent of α for which there exist γ solving the inequalities. In this case any γ from the interval $(0, \frac{2-\alpha}{4})$ is a solution. For our purposes we have taken $\gamma = \frac{2-\alpha}{5}$.

4 Return from $-\infty$ and deviations from the deterministic trajectory: exit from unbounded interval

With the aim of proving an analogue of Proposition 2.1, we study in this section the exit problem of the solution of our stochastic differential equation from the unbounded interval $J = (-\infty, a]$. In this case we shall use the condition that U' increases faster than a linear function at $-\infty$ which guarantees a return from infinity in finite time for the unperturbed deterministic motion. For simplicity we assume that for large $|x|$, U is a power function, i.e. $|U(x)| = c_1|x|^{2+c_2}$, $c_1, c_2 > 0$ for $|x| \geq N$. This condition can be weakened (see Remark 4.1 at the end of this section). We stick to it to avoid technicalities irrelevant for the main aim of the paper.

Fix two more positive numbers r and R such that $N < r < R$, and such that with $T_R = \int_{-\infty}^{-R+10} \frac{dy}{|U'(y)|}$ we have $-r = Y_{T_R}(-R)$. Consider the equations (27) on the unbounded interval J. As in section 2 we estimate on the basis of the representations $x^\varepsilon(x) = Y_t(x) + \varepsilon Z_t^\varepsilon(x) + R_t^\varepsilon(x)$, with $Y_t(x) = x - \int_0^t U'(Y_s(x))\,ds$ and $Z_t^\varepsilon(x) = \xi_t^\varepsilon - \int_0^t \xi_{s-}^\varepsilon U''(Y_s(x))\frac{U'(Y_t(x))}{U'(Y_s(x))}\,ds$. We first prove an estimate enabling us to transfer Lemma 2.2 to unbounded intervals.

Lemma 4.1 *The inequality* $\sup_{t\in[0,T_R]} |Z_t^\varepsilon(x)| \leq 2\sup_{t\in[0,T_R]} |\xi_t^\varepsilon|$ *holds a.s. for* $x \leq -R$.

Proof: For all $x \leq -R$ by definition of R and r we have $Y_t(x) \leq -r$. Moreover, by assumption $U''(Y_t(x)) > 0$ for $t \in [0, T_R]$, whence

$$Z_t^\varepsilon(x) \leq \sup_{t\in[0,T_R]} |\xi_t^\varepsilon| \left(1 + \sup_{x\leq -R} \sup_{t\in[0,T_R]} \int_0^{T_R} U''(Y_s(x)) \frac{U'(Y_t(x))}{U'(Y_s(x))}\,ds\right). \tag{95}$$

We show that the integral in the latter parenthesis is uniformly bounded in x. Denote $Y_t(x) = v$. Then $dv = \dot{Y}_t(x)\,dt = -U'(Y_t(x))\,dt = -U'(v)\,dt$. Therefore

$$\begin{aligned} \int_0^{T_R} U''(Y_s(x)) \frac{U'(Y_t(x))}{U'(Y_s(x))}\,ds &= -U'(Y_{T_R}(x)) \int_x^{Y_{T_R}(x)} \frac{U''(v)}{U'(v)^2}\,dv \\ &= U'(Y_{T_R}(x)) \left[\frac{1}{U'(Y_{T_R}(x))} - \frac{1}{U'(x)}\right] \leq 1. \end{aligned}$$

■

Here is the analogue of Lemma 2.2.

Lemma 4.2 *There is a constant* $C_Z' > 0$ *such that the inequality*

$$\sup_{t\in[0,T]} |Z_t^\varepsilon(x)| \leq C_Z' \sup_{t\in[0,T]} |\xi_t^\varepsilon| \tag{96}$$

holds a.s. for $x \leq a$, $T > 0, \varepsilon > 0$.

Proof: The proof obviously has to combine the previous Lemma with Lemma 2.2 on bounded intervals. The inequality holds with $C'_Z = 1 + C_Z$. ■

To estimate deviations of the random paths from the paths of the deterministic equation, we restrict from the start to sets of bounded scaled noise.

Lemma 4.3 *On the event* $\{\sup_{t\in[0,T_R]} |\varepsilon\xi^\varepsilon_t| < 1\}$ *the following inequality holds a.s.*

$$\sup_{t\in[0,T_R]} |x^\varepsilon_t - Y_t(x)| < 10 \tag{97}$$

uniformly for $x \leq -R$.

Proof: It follows from Lemma 4.1, that $\sup_{t\in[0,T_R]} |\varepsilon Z^\varepsilon_t(x)| < 2$ a.s. on the event $\{\sup_{t\in[0,T_R]} |\varepsilon\xi^\varepsilon_t| < 1\}$. We show that the rest term $|R^\varepsilon|$ is bounded by 8. Indeed, the rest term satisfies the integral equation

$$R^\varepsilon_t(x) = \int_0^t \left[-U'(Y_s(x) + \varepsilon Z^\varepsilon_{s-}(x) + R^\varepsilon_s(x)) + U'(Y_s(x)) + U''(Y_s(x))\varepsilon Z^\varepsilon_{s-}(x)\right] ds. \tag{98}$$

$R^\varepsilon(x)$ is absolutely continuous a.s. and $R^\varepsilon_0(x) = 0$. Assume, that there exists a smallest $\tau \in [0, T_R]$ such that $R^\varepsilon_\tau(x) = 8$. Then the left Dini derivative of $R^\varepsilon(x)$ at this point is non-negative, i.e.

$$-U'(Y_\tau(x) + \varepsilon Z^\varepsilon_{\tau-}(x) + 8) + U'(Y_\tau(x)) + U''(Y_\tau(x))\varepsilon Z^\varepsilon_{\tau-}(x) \geq 0. \tag{99}$$

On the other hand, our conditions on U guarantee

$$\begin{aligned} &- U'(Y_\tau(x) + \varepsilon Z^\varepsilon_{\tau-}(x) + 8) + U'(Y_\tau(x)) + U''(Y_\tau(x))\varepsilon Z^\varepsilon_{\tau-}(x) \\ &\qquad < -U'(Y_\tau(x) + 6) + U'(Y_\tau(x)) + 2U''(Y_\tau(x)) < 0, \end{aligned} \tag{100}$$

and a contradiction is reached. The estimate $R^\varepsilon_t(x) > -8$ is obtained analogously. ■

The following lemma is an analogue of Lemma 2.3 and gives a rough estimate for the remainder term R^ε.

Lemma 4.4 *There exists* $C > 0$ *such that for* $x \in [-R, a]$, $T > 0$

$$\sup_{t\in[0,T]} |R^\varepsilon_t| < C \tag{101}$$

a.s. on the event $\{\sup_{t\in[0,T]} |\varepsilon\xi^\varepsilon_t| < 1\}$.

Lemma 4.4 again has localizing consequences: It states precisely that the solution process x^ε, with initial state confined to $[-R, a]$, stays bounded by a deterministic constant on sets of the form $\{\sup_{t\in[0,T]} |\varepsilon\xi^\varepsilon_t(\omega)| < 1\}$. Therefore, in the small noise limit, only local properties of U are relevant to our analysis.

Let us paraphrase the most important aspect of what we found in the previous Lemmas due to finite return from $-\infty$. With probability close to one, the random trajectory starting at $x \leq -R$ reaches the finite interval $[-R, -r]$ in a finite non-random time T_R which does not depend on ε. Our investigation therefore reduces to the study of the dynamics of paths starting in the finite interval $[-R, a]$. Since the deterministic trajectories starting in $[-R, a]$ do not leave this interval, the statement of Lemma 2.2 does not change. Due to Lemma 4.4 the estimate of the rest term R^ε given in Lemma 2.5 also holds unchanged.

Thus, in the case of the unbounded interval we have all necessary tools to estimate the exit probabilities.

Remark 4.1 The conditions on the behavior of the potential U at $-\infty$ can be weakened. Indeed, a slight extension of the proof of Lemma 4.1 allows to drop the convexity condition $U'' > 0$. Furthermore, to show that R^ε is bounded from above by some $p_1 > 0$, we need to guarantee that the integrand in (98) is negative for $R^\varepsilon_\tau = p_1$ under the condition $|\varepsilon Z^\varepsilon_{\tau-}| \leq p_2$, $p_2 > 0$. This leads to the inequality $-\inf_{\theta\in[p_2-p_1,p_2+p_1]} U'(y+\theta) + U'(y) + p_1|U''(y)| < 0$, which has to hold for $y \leq -N$. For instance, for the power function $U(x) = c_1|x|^{2+c_2}$ considered above this inequality is equivalent to $-(p_2 - 2p_1)|y|^{-1} + \mathcal{O}(|y|^{-2}) < 0$, $y \to -\infty$, and thus holds for any $p_2 > 2p_1 > 0$.

5 The law of $\tau(\varepsilon)$

In this section we estimate $\mathbf{P}_x(\tau(\varepsilon) > u)$ for $u \geq 0$ as $\varepsilon \to 0$. Indeed, the extensions of propositions 3.1 for the estimation above and 3.2 for lower bounds with parameter $b = +\infty$ take the following form.

Proposition 5.1 *Let $\delta = \min\{\alpha/2, \gamma/2\}$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $x \in (-\infty, a - \varepsilon^\gamma]$ and $u \geq 0$*

$$\mathbf{P}_x(\tau(\varepsilon) > u) \leq \exp\left\{-u\frac{\varepsilon^\alpha}{\alpha a^\alpha}(1 - C\varepsilon^\delta)\right\}(1 + C\varepsilon^\delta). \tag{102}$$

Proposition 5.2 *Let $\delta < \min\{\alpha/2, \gamma/2\}$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $x \in (-\infty, a - \varepsilon^\gamma]$ and $u \geq 0$*

$$\mathbf{P}_x(\tau(\varepsilon) > u) \leq \exp\left\{-u\frac{\varepsilon^\alpha}{\alpha a^\alpha}(1 + C\varepsilon^\delta)\right\}(1 - C\varepsilon^\delta). \tag{103}$$

Proof: The arguments to prove the estimates are similar to those of the bounded case. We just need to adapt Steps 1, 2, 3 and 4 (see sections 3.1 and 3.2) to the case of an unbounded interval. Let us consider for example the extension of Step 1 from section 3.1. The basic formula (72) holds in the case of unbounded intervals with J replacing I. We demonstrate how to modify the reasoning just in the series of inequalities (74) and in the estimate (78). The other estimates are obtained analogously.

Firstly, we estimate $\mathbb{I}\{x_s^1(y) \in J, s \in [0, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\}$ for $-R \leq y \leq a$. Denote $A = \{\sup_{t\in[0,T_1]} |\varepsilon\xi_t^\varepsilon| < 1\}$. On the event A, the trajectory $x_t^1(y)$, $t \in [0, T_1]$, belongs to a compact interval, so its dynamics is indistinguishable form the one treated in the bounded case. Therefore, we have

$$\begin{aligned}
&\mathbb{I}\{x_s^1(y) \in J, s \in [0, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\} \\
&= \mathbb{I}\{x_s^1(y) \in J, s \in [0, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\}\,(\mathbb{I}\{A\} + \mathbb{I}\{A^c\}) \\
&\leq \mathbb{I}\{x_s^1(y) \in J, s \in [0, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J, A\} + \mathbb{I}\{A^c\} \leq \cdots \\
&\leq \mathbb{I}\{\sup_{[0,T_1]} |x_s^1(y) - Y_s(y)| > \tfrac{\varepsilon^\gamma}{2}\} + \mathbb{I}\{\varepsilon W_1 \in J_{\varepsilon^\gamma}^+\} + \mathbb{I}\{|\varepsilon W_1| > \tfrac{\varepsilon^\gamma}{2}, T_1 < \mu_1|\ln\varepsilon|\} + \mathbb{I}\{A^c\}
\end{aligned} \tag{104}$$

The case $y \leq -R$ is slightly more complicated since we have to treat the return of $x^1(y)$ to a compact interval in a finite time. Denote

$$\begin{aligned}
B = \{\omega : \sup_{[0,T_R\wedge T_1]} |\varepsilon\xi_t^\varepsilon| < 1\} \cap \{\omega : \sup_{[T_R\wedge T_1, T_1]} |\varepsilon\xi_t^\varepsilon - \varepsilon\xi_{T_R\wedge T_1}^\varepsilon| < 1\} \\
\supseteq \{\omega : \sup_{[0,T_1]} |\varepsilon\xi_t^\varepsilon| < 1\} = A.
\end{aligned} \tag{105}$$

Then we have

$$\begin{aligned}
&\mathbb{I}\{x_s^1(y) \in J, s \in [0, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\} \\
&\leq \mathbb{I}\{x_s^1(y) \in J, s \in [0, T_R \wedge T_1], x_s^1(y) \in J, s \in [T_R \wedge T_1, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\} \cap B + \mathbb{I}\{A^c\} \\
&\leq \mathbb{I}\{x_s^1(y) \in J, s \in [T_R \wedge T_1, T_1], x_{T_1}^1(y) + \varepsilon W_1 \in J\} \cap B + \mathbb{I}\{A^c\} \leq \cdots \\
&\leq \mathbb{I}\{\sup_{s\in[T_R\wedge T_1, T_1]} |x_s^1(y) - Y_{s-T_R\wedge T_1}(x_{T_R\wedge T_1}^1(y))| > \tfrac{\varepsilon^\gamma}{2}\} + \mathbb{I}\{\varepsilon W_1 \in J_{\varepsilon^\gamma}^+\} \\
&+ \mathbb{I}\{|\varepsilon W_1| > \tfrac{\varepsilon^\gamma}{2}, T_R \wedge T_1 < \mu_1|\ln\varepsilon|\} + \mathbb{I}\{A^c\} \\
&\leq \sup_{y\in[-R,-r]} \mathbb{I}\{\sup_{s\in[0,T_1]} |x_s^1(y) - Y_s(y)| > \tfrac{\varepsilon^\gamma}{2}\} + \mathbb{I}\{\varepsilon W_1 \in J_{\varepsilon^\gamma}^+\} \\
&+ \mathbb{I}\{|\varepsilon W_1| > \tfrac{\varepsilon^\gamma}{2}, T_1 < \mu_1|\ln\varepsilon|\} + \mathbb{I}\{A^c\}.
\end{aligned} \tag{106}$$

These estimates may be treated in a way similar to (78). In fact,

$$\mathbf{E}\left[\sup_{y\in J} \mathbb{I}\{x_s^1(y) \in J, s \in (0, T_1], x_{T_1}^1 + \varepsilon W_1 \in J\}\right] \leq 1 - \varepsilon^{\alpha/2}\frac{1}{2a^\alpha}(1 - C\varepsilon^\delta) \tag{107}$$

for some $C > 0$. The other steps are modified analogously. ■

Proof of Theorem 1.2: Combine the estimates of the above Propositions. ■

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