Partial Equilibrium and Market Completion *

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Abstract

We consider financial markets with agents exposed to an external source of risk which cannot be hedged through investments on the capital market alone. The sources of risk we think of may be weather and climate. Therefore we face a typical example of an incomplete financial market. We design a model of a market on which the external risk becomes tradable. In a first step we complete the market by introducing an extra security which valuates the external risk through a process parameter describing its market price. If this parameter is fixed, risk has a price and every agent can maximize the expected exponential utility with individual risk aversion obtained from his risk exposure on the one hand and his investment into the financial market consisting of an exogenous set of stocks and the insurance asset on the other hand. In the second step, the market price of risk parameter has to be determined by a partial equilibrium condition which just expresses the fact that in equilibrium the market is cleared of the second security. This choice of market price of risk is performed in the framework of nonlinear backwards stochastic differential equations.

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Introduction

In recent years, a new type of financial products on incomplete markets has appeared at the interface of finance and insurance. Its purpose is *securitization*, i.e. to shift certain (re-) insurance risks to the capital markets. The products we focus on deal with natural exterior risks generated in particular by weather and climate phenomena. The first product of this type was traded by the New York and the Chicago Mercantile Exchange and was called the *New York HDD swap* (see [Dav]). The demand for these products may come from energy companies supplying gas to retail distributors. If for example the heating season in winter is unusually warm, due to a smaller volume of gas sold profits may shrink. Another example is given by the risk a reinsurance company faces due to big accumulative losses for example in farming or fishing caused by the most well known short term climate event of the El Niño Southern Oscillation (ENSO).

Techniques for pricing and hedging derivatives on incomplete markets have been developing fast during recent years. They include utility indifference arguments (see for example Delbaen et al. [Del] for a more stochastic approach or Musiela, Zariphopoulou [MZ] for an approach more in the analytical spirit of the HJB formalism), mean variance hedging (see the survey by Schweizer [Sch]), backwards stochastic differential equations (BSDE) (see El Karoui et al. [EPQ]), more generally stochastic control theory methods (see the recent thesis by Barrieu [Bar]) or the shadow price approach (see Davis [Dav]). First applications of these techniques to the special setting of weather derivatives were given. In Becherer's thesis [Bec] for special jump type weather risks utility indifference arguments yield surprisingly explicit descriptions of optimal hedges. Control theoretic methods are applied in the thesis of Barrieu [Bar], in the setting of incomplete markets as well, to find strategies for insuring exterior risks by optimal choice of derivatives. A survey on the method of security design in market models can be found in [DR]. Techniques of market completion play a role for example in [F], [W]. There is a vast literature on mathematical problems in reinsurance. We only mention [M], [Ber], [GY], and refer to the survey article [Aa].

This paper proposes a simple model that explains how non–financial external risks can be traded and priced. Our approach is based on the two concepts of *market completion* and *partial equilibrium*. The equilibrium is attained on a market which contains finitely many agents interested in trading the external risk. They may for example represent companies like insurance companies with profits under the influence of an external risk source such as weather, or reinsurance companies running even bigger risks caused by extreme weather or climate events, or just risk takers who want to diversify their portfolios. In our model the profits of the companies are given by a risky income that depends both on their external risk exposure and on the economic development. This income is realized at a terminal time T. The economic component of the risk alone can be hedged using an exogenously given stock market. But for the external risk with an independent source of uncertainty the situation is different. For every agent who has to face it the stock market is consequently incomplete. And here the first concept of our approach enters the stage. In order to make external risk tradable, we *complete the market*. To this end we construct a second security which we could call an insurance asset. It is traded among the finitely many market participants who shift and relocate their individual risks created by the external source. Its price process is characterized by the stipulation that the overall offer and demand for this security are equal. And this is the second key concept of our approach: the partial market clearing condition concerning only the security through which external risk becomes tradable creates a *partial equilibrium*. The market price of risk for the insurance asset is adjusted in such a way that the market is cleared of this security. Trading in the other component of the market, the stock exchange, is not subject to constraints. Market clearing for the stock is not required.

These basic assumptions suggest the following algorithm for finding the appropriate price of risk in the market. Suppose first that a possible candidate θ for this price of external risk is given. The agents' incentives to act are steered by their preferences. In our model they are described by their expected utility following an exponential utility function with individual risk aversion coefficient. According to the model assumptions explained above, their budget set consists of two components: a payoff that is attainable by investing into stock and insurance asset given an initial capital, plus the random income due to their external risk exposure. Given the stock price process, the candidate θ for the market price of risk, and the individual incomes due to risk exposure, every agent will choose an individual investment strategy into the two types of assets which maximizes the payoff in his budget set for his utility concept.

Next, among all possible prices of risk, and given the individual optimal investment strategies of each agent, we have to choose the one which satisfies the equilibrium (market clearing) condition. This price of risk θ^* is uniquely determined, and can be described by means of a non-linear BSDE. Once it is available, it is possible to find the market value of the payoffs describing the exposures to external risk. It is given by the initial capital of a trading strategy that replicates this payoff for every individual agent. Alternatively, it is given by the expectation of the risky incomes with respect to the unique martingale measure Q^{θ^*} for the stock price process and the second security. This pricing rule is linear.

Hence in this model the external risk dynamically determines in a unique way the market price of risk of the second security via the risky incomes, the preferences and the partial market clearing condition. The dynamics is created in the following way: if the agents start trading at a time $\tau \in (0, T)$, the price process of the second security is unchanged.

The idea of market completion with a partial equilibrium just sketched can be reinterpreted to also apply in the following more general situation in which more than one asset is needed to complete the market. In this case *completion* is attained in a different way. As before, agents are allowed to freely trade at the stock market. In addition, they may trade the external risk by buying and selling random payoffs among themselves which they are able to choose freely. Random payoffs are priced using one *pricing rule*. The value of a payoff which is replicable by a trading strategy must equal the initial capital of this strategy. We therefore use pricing rules which are linear functions of the payoffs and which can be described as expectations under probability measures equivalent to the real world measure P. The condition to be consistent with the stock price in addition leads to those equivalent probability measures for which the stock price process is a martingale. In the first version of our model market completion was achieved by choosing a second security parameterized by a market prize of risk process θ which generates a martingale under a unique probability measure Q^θ. In this version linear pricing rules Q^θ parametrized by these market prices of external risk describe the class of relevant martingale measures for the stock price.

Given an admissible pricing rule Q^θ, every agent chooses in his budget set the payoff which maximizes his expected utility. In this setting, the market clearing condition leading to the *partial equilibrium* reads as follows. The difference between the sum of the incomes of the agents due to external risk exposure and the sum of the preferred payoffs viewed with particular linear pricing rules must be replicable by a trading strategy based on the stock alone. The linear pricing rule Q^{θ^*} obtained this way is uniquely determined in a dynamic way: if the agents start trading at time $\tau \in (0, T)$, the partial equilibrium is attained by the same pricing rule.

To achieve the goals just sketched, we apply utility maximization techniques for complete markets using martingale and BSDE methods. Martingale methods are treated in [KLS], [CH] and [Pl]. The construction of a unique equilibrium in a Brownian filtration is given in [KLS2], where market participants obtain an endowment of a perishable good at a positive rate. They maximize the expected utility of consumption in the trading interval. The stock price process and the interest rate of the bond are constructed in such a way that offer and demand of the stocks and the bond are equal (zero net supply) and the consumption of the perishable good is equal to the endowment. For the use of BSDE methods in control theoretic problems of the kind we encounter see Barrieu [Bar].

The paper is organized as follows. In section 1 the background for our model is explained: the price processes X^S, X^E, wealth processes and trading strategies are given a detailed treatment. In section 2 we recall the solution of the utility maximization problem for the individual agents. Section 3 is devoted to the discussion of the partial equilibrium, by means of BSDE methods. In section 4, in a more general framework, the alternative construction of the partial equilibrium based on pricing measures is carried out.

In a companion paper [CIM], the simple model presented here is applied to climate risk, a particularly interesting external risk source. Numerical methods are developed based on the well known correspondence between non-linear BSDE and viscosity solutions of quasi-linear PDE to simulate optimal wealth and strategies of individual agents participating in the market. We focus on two or three agents exposed to the climate phenomenon of ENSO.

1 Market completion, price processes and trading strategies

The mathematical frame is given by a probability space $(\Omega, \mathcal{F}, P)$ carrying a two–dimensional Brownian motion $W = (W_1, W_2)$ indexed by the time interval $[0, T]$, where $T > 0$ is a deterministic time horizon. Note here that stochastic processes indexed by $[0, T]$ will be written $X = (X_t)_{t\in[0,T]}$. The filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the completion of the natural filtration of W.

We shall use the following notations. Let Q be a probability measure on $\mathbf{F}$, $k \in \mathbf{N}, p \geq 1$. Then $L^p(Q)$ stands for the set of equivalence classes of Q–a.s. equal $\mathcal{F}_T$–measurable random variables which are p–integrable with respect to Q. $\mathcal{H}^k(Q, \mathbf{R}^d)$ denotes the set of all $\mathbf{R}^d$–valued stochastic processes ϑ that are predictable with respect to $\mathbf{F}$ and such that $E^Q[\int_0^T \|\vartheta_t\|^k dt] < \infty$. Here and in the sequel E^Q denotes the expectation with respect to Q. We write l for the Lebesgue measure on $[0, T]$ or $\mathbf{R}$. $\mathcal{H}^\infty(Q, \mathbf{R}^d)$ is the set of all $\mathbf{F}$–predictable $\mathbf{R}^d$–valued processes that are $l \otimes Q$–a.e. bounded on $[0, T] \times \Omega$.

For a continuous semimartingale M with quadratic variation $\langle M\rangle$ the stochastic exponential $\mathcal{E}(M)$ for an adapted continuous stochastic process M is given by

$$\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t), \qquad t \in [0, T].$$

Let us now explain the first version of our model in more formal details.

1.1 Stock market structure

The stock market is represented by an exogenous $\mathbf{F}$–adapted index or stock price process X^S indexed by the trading interval $[0, T]$. The dynamics of this price process evolves according to the stochastic integral equation

$$X_t^S = X_0^S + \int_0^t X_s^S (b_s^S \, ds + \sigma_s^S \, dW_s^1), \quad t \in [0, T],$$

where X_0^S is a positive constant, so that we have

$$X_t^S = X_0^S \, \mathcal{E}\left(\int (b_s^S ds + \sigma_s^S \, dW_s^1) \right)_t . \tag{1}$$

Throughout the paper we shall work with the following assumption concerning the drift b^S and volatility σ^S of the stock price process X^S :

Assumption 1.1
$b^S \in \mathcal{H}^\infty(P, \mathbf{R})$,
$\sigma^S \in \mathcal{H}^\infty(P, \mathbf{R})$,
there is $\varepsilon > 0$ *such that* $\sigma^S > \varepsilon$.

Observe that due to this assumption the process

$$\theta^S := \frac{b^S}{\sigma^S} \tag{2}$$

is also contained in $\mathcal{H}^\infty(P, \mathbf{R})$ and $P[X_t^S > 0 \text{ for all}\, t \in [0, T]] = 1$.

1.2 Agents subject to external risk

The external risk component enters our model through an $\mathbf{F}$–adapted stochastic process K, indexed by the trading interval as well. As an example, one might think of a climate process, such as the temperature process in the Eastern South Pacific which gives rise to the climate phenomenon of ENSO which largely affects the national economies of the neighboring states. See [CIM], where the effects of this phenomenon and risk transfer strategies based the concepts of which are developed in this paper are captured by numerical simulations.

Agents on the market are symbolized by the elements a of a finite set I. They can use a bond with interest rate zero. Every agent $a \in I$ is supposed to be endowed with an initial capital $v_0^a \geq 0$. At the end of the trading interval at time T he receives a stochastic income H^a which describes the profits that this agent or the company he represents obtains from his usual business. The income H^a is supposed to be a real valued bounded $\mathcal{F}_T$–measurable random variable function of the processes X^S and K, i.e.

$$H^a = g^a(X^S, K).$$

For example, if $\mathbf{D}$ denotes the space of adapted stochastic processes that are right continuous with left limits, we could have K possessing trajectories in $\mathbf{D}$ and g^a a real valued bounded function defined on $\mathbf{D} \times \mathbf{D}$. A typical example covered by these assumptions is the following. Think of two agents, say a company c and a bank b. c could for example possess an income $H^c = g^c(K)$ purely dependent on the exterior risk. The bank has an income $H^b = g^b(X^S)$ which only depends on the stock market. c wants to hedge fluctuations caused by the external factor and signs a contract with b to transfer part of this risk. b's interest in the contract could be based on the wish to diversify its portfolio. For concrete numerically investigated toy examples in the context of ENSO risks see [CIM].

1.3 Market completion

In order to complete the market, we want to construct a second security through which external risk can be traded with price process X^E of a form given by the following stochastic integral equation

$$X_t^E = X_0^E + \int_0^t X_s^E (b_s^E ds + \sigma_s^E dW_s^2), \quad t \in [0, T], \tag{3}$$

with coefficient processes b^E and $\sigma^E \in \mathcal{H}^2(P, \mathbf{R})$, and such that for some $\varepsilon > 0$ we have $\sigma^E > \varepsilon$. Let

$$\theta^E := \frac{b^E}{\sigma^E}. \tag{4}$$

The processes θ^S, θ^E are called market price of risk of the stock and the insurance security. Every market price of risk θ^E of the second security is supposed to belong to the following set:

$$\mathcal{K}_2 = \left\{ \theta^E \in \mathcal{H}^2(P, \mathbf{R}) \,\middle|\, \int_0^{\cdot} \theta_s^E dW_s^2 \;\text{ is a } (P, \mathbf{F}) - \text{BMO martingale} \right\}.$$

The market price of risk vector θ at the same time parametrizes a class of probability measures Q^θ for which the price processes (X^S, X^E) are martingales. More formally, denote

$$X := \begin{pmatrix} X^S \\ X^E \end{pmatrix}, \quad \theta := \begin{pmatrix} \theta^S \\ \theta^E \end{pmatrix} \quad \text{and} \quad \sigma := \begin{pmatrix} \sigma^S & 0 \\ 0 & \sigma^E \end{pmatrix}. \tag{5}$$

The matrix valued process σ is invertible for all $t \in [0,T]$ P–a.s. With $\theta^E \in \mathcal{K}_2$ and θ^S according to Assumption 1.1 it is seen by using (32) (Appendix) that the process $(\int_0^t \theta_s dW_s)_{t\in[0,T]}$ is a P–BMO martingale. This property in turn guarantees that the change of measure obtained by drifting W by θ induces an equivalent probability.

Lemma 1.2 *Suppose that $\theta = (\theta^S, \theta^E)$ with θ^S satisfying Assumption 1.1 and $\theta^E \in \mathcal{K}_2$. Then the process $Z^\theta := \mathcal{E}(-\int_0^\cdot \theta_t dW_t)$ defines the density process of an equivalent change of probability.*

Proof The process Z^θ is the stochastic exponential of a BMO–martingale. By Theorem 2.3 in [Kaz] it is a uniformly integrable $(P, \mathbf{F})$–martingale.

□

According to Lemma 1.2 we may define the measure Q^θ with Radon–Nikodym density with respect to P given by

$$\frac{dQ^\theta}{dP} = Z_T^\theta = \mathcal{E}\left(-\int_0^T \theta_t dW_t\right)_T = \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \|\theta_t\|^2 dt\right). \tag{6}$$

This provides the unique probability for which the price process $X = (X^S, X^E)$ given by (1) and (3) is a martingale. Hence the choice of a particular insurance asset completing the market leads to a class of equivalent martingale measures for the price dynamics parametrized by the price of risk processes. By the well known Lévy characterization $W^\theta = W + \int_0^\cdot \theta_s ds$ is a Q^θ–Brownian motion.

1.4 Trading and wealth

The market being equipped with this structure, each agent $a \in I$ will maximize the terminal wealth obtained from his portfolio in the securities (X^S, X^E) and his random risky income subject to the exterior risk H^a, according to his individual preferences. Thereby he will be allowed to follow trading strategies to be specified in the following. A trading strategy is given by a 2–dimensional $\mathbf{F}$–predictable process $\pi = (\pi_t)_{0\le t\le T}$ such that $\int_0^T \|\pi_t\sigma_t\|^2 dt < \infty$ P–a.s., hence $\int_0^\cdot (\frac{\pi_{1,s}}{X_s^S}, \frac{\pi_{2,s}}{X_s^E}) dX_s$ is well–defined. This notation of a trading strategy describes the number of currency units invested in each security. The wealth process $V = V(\pi) = V(c,\pi)$ of a trading strategy π with initial capital c is given by

$$V_t = c + \int_0^t \left(\frac{\pi_{1,s}}{X_s^S}, \frac{\pi_{2,s}}{X_s^E}\right) d\begin{pmatrix} X_s^S \\ X_s^E \end{pmatrix}, \quad t \in [0,T].$$

The number of shares of security i is $\frac{\pi_{i,t}}{X_t^i}$, $i = S, E$. For the ease of notation, we shall write in the sequel $\frac{dX}{X}$ for the vector increment $(\frac{dX^S}{X^S}, \frac{dX^E}{X^E})$. Trading strategies are self–financing. This means that those parts of the wealth not invested into X^S or X^E are kept in the bond. Gains or losses are only caused by trading with the securities. The wealth process can equivalently be written as

$$V_t(c,\pi) = c + \int_0^t \pi_s \sigma_s (dW_s + \theta_s ds) = c + \int_0^t \pi_s \sigma_s dW_s^\theta, \quad t \in [0,T]. \tag{7}$$

A set Φ of strategies is called free of arbitrage if there exists no trading strategy $\pi \in \Phi$ such that

$$V_0(\pi) = 0, \quad V_T(\pi) \geq 0 \quad \text{and} \quad P[V_T(\pi) > 0] > 0.$$

We have to restrict the set of trading strategies by defining the set of admissible strategies in order to exclude opportunities of arbitrage.

Definition 1.3 (Admissible Strategies) *The set of admissible trading strategies $\mathcal{A}$ is given by the collection of the 2–dimensional predictable processes π with $\int_0^T \|\pi_t \sigma_t\|^2 dt < \infty$ Q^θ–a.s. such that the wealth process $V(c,\pi)$ is a $(Q^\theta, \mathbf{F})$–supermartingale.*

The set of admissible strategies $\mathcal{A}$ is free of arbitrage. In fact, we get from $V_0(0,\pi) = 0$ and $V_T(0,\pi) \geq 0$ that $V_T(0,\pi) = 0$ Q^θ– and thus P– a.s. Examples are strategies π with initial capital v_0 such that $V(v_0,\pi)$ is bounded from below uniformly on $[0,T]\times\Omega$. In this case, $V(v_0,\pi)$ is a local Q^θ– martingale bounded from below, hence a Q^θ–supermartingale.

2 Utility maximization

Fixing a particular market price of risk $\theta^E \in \mathcal{K}_2$, in this section we describe the individual behavior of an agent $a \in I$. In particular, the impact of the choice of θ^E determining the price process X^E of the insurance asset on his terminal wealth and trading strategy is clarified. Let us emphasize at this point that the introduction of X^E completes the market with price process X having components X^S and X^E. We use well known results about utility maximizing trading strategies and the associated terminal wealth in a complete market. They can be found e.g. in [KLS] for the maximization of an expected utility and in [Am] for the optimization of the conditional expected utility with respect to a non trivial sigma algebra.

Every agent $a \in I$ has initial capital v_0^a at his disposal. At the terminal time T he receives a random income possibly depending on external risk and described by an $\mathcal{F}_T$–measurable bounded random variable H^a. The investor wants to hedge fluctuations in his income H^a or diversify his portfolio. His preferences are described by the expected utility using the utility function

$$U^a(x) = -\exp(-\alpha_a x) \qquad x \in \mathbf{R},$$

with an individual risk aversion coefficient $\alpha_a > 0$. The agents act as price takers and are not able to change X^S and X^E.

2.1 Maximization for start at time 0

The individual utility maximization problem for the traders acting on the whole time interval $[0, T]$ then takes the following mathematical form. Each one of them wants to find a trading strategy $\pi^a \in \mathcal{A}$ which attains

Problem 2.1 (Individual utility maximization, start at 0)

$$\begin{aligned} J^a(v_0^a, H^a, X^S, X^E) &= \sup_{\pi \in \mathcal{A}} E[-\exp(-\alpha_a(V_T(v_0^a, \pi) + H^a)] \\ &= \sup_{\pi \in \mathcal{A}} E\left[-\exp\left(-\alpha_a\left(v_0^a + \int_0^T \pi_s \frac{dX_s}{X_s} + H^a\right)\right)\right]. \end{aligned}$$

Since $x \mapsto -\exp(-\alpha x)$ is bounded from above, the expectations appearing in Problem 2.1 are well defined. It will be more convenient to reformulate our utility maximization problem using the martingale measure Q^θ with Brownian motion W^θ of our price process $X = (X^S, X^E)$. In particular, we aim for an alternative description of the budget set, described above as the set of final claims attained by admissible trading strategies, in terms of the martingale measure. This will turn out to be important in section 4 where we generalize our model to more complex situations: martingale measures will correspond to pricing rules there. At the end of the trading period, every agent has a claim of $B = V_T(v_0^a, \pi) + H^a$ based on his initial capital, his investments in X and external risk exposure. On the one hand, $V(v_0^a, \pi)$ being a $Q^\theta-$ supermartingale for each admissible trading strategy π this claim has to satisfy the inequality $E^\theta(B) \leq v_0^a + E^\theta(H^a)$. If it is even a $Q^\theta-$martingale, equality holds. On the other hand, the market being complete, every claim of this type can be replicated by appealing to the martingale representation theorem with respect to the Brownian motion W^θ under Q^θ. More precisely, H^a being bounded, for any $B \in L^1(Q^\theta)$ we may find an $\mathbf{F}-$predictable process ϕ satisfying $\int_0^T \|\phi_s\|^2 ds < \infty$ Q^θ–a.s. and

$$\begin{aligned} B - H^a &= E^\theta[B - H^a] + \int_0^T \phi_s dW_s^\theta \\ &= v_0^a + \int_0^T \phi_s \sigma_s^{-1} \frac{dX_s}{X_s} \\ &= V_T(v_0^a, \phi\,\sigma^{-1}). \end{aligned}$$

So we may set

$$\pi = \phi\,\sigma^{-1} \tag{8}$$

to obtain an admissible strategy. Here σ is defined by (5).

To summarize the result of our arguments in a slightly different manner: a random variable $B \in L^1(Q^\theta, \mathcal{F}_T)$ is the sum of the terminal value of the wealth process of an admissible trading strategy π with initial capital v_0 and a terminal income H^a if and only if $E^\theta[B] = v_0 + E^\theta[H^a]$.

This implies that our problem (2.1) boils down to the following maximization problem over random variables given by the claims. We collect claims B composed of final wealths of admissible strategies and final incomes H^a in the *budget set*

$$\mathcal{B}(v_0, H^a, \theta^S, \theta^E) := \{D \in L^1(Q^\theta, \mathcal{F}_T) \ : \ E^\theta[D] \leq v_0 + E^\theta[H^a]\}, \tag{9}$$

and then have to find the random variable $D^a(\theta^S, \theta^B)$ that attains

$$J^a(v_0^a, H^a, \theta^S, \theta^E) := \sup_{D \in \mathcal{B}(v_0^a, H^a, \theta^S, \theta^E)} E[-\exp(-\alpha_a D)]. \tag{10}$$

The solution is obtained by well known methods via an application of the Fenchel–Legendre transform to the concave function $x \mapsto -\exp(-\alpha_a x)$.

Theorem 2.2 *Let H^a be a bounded $\mathbf{F}_T$–measurable random variable, $v_0^a \geq 0$. Define*

$$D^a(\theta^S, \theta^E) := D^a(v_0^a, H^a, \theta^S, \theta^E) = -\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta)$$

where λ_a is the unique real number such that

$$E^\theta[-\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta)] = v_0^a + E^\theta[H^a].$$

Then $D^a(\theta^S, \theta^E)$ is the solution of the utility maximization problem (10)for agent $a \in I$.

Proof The main body of the proof is given by Theorem 2.3.2 of [KLS2], stated for utility functions satisfying the Inada conditions, i.e. $U'(\infty) = 0$, $U'(0+) = \infty$, and under the hypothesis that the quadratic variation of $\int_0^\cdot \theta_s dW_s$ is bounded. In our setting, this process is a BMO–martingale for which the quadratic variation is not necessarily bounded. Therefore we have to show that for every $a \in I, v \in \mathbf{R}$ there exists $\lambda_a > 0$ satisfying

$$E^\theta[-\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta)] = v. \tag{11}$$

A sufficient condition for this is that the relative entropy of Q^θ with respect to P is finite. We recall that for probability measures Q, R on $\mathbf{F}$ the relative entropy of Q with respect to R is defined by

$$H(Q|R) = \begin{cases} E^Q[\log \frac{dQ}{dR}], & \text{if} \quad Q \ll R, \\ \infty, & \text{if} \quad \text{not}. \end{cases}$$

Therefore we may finish the proof of the Theorem with an application of the following Lemma, stated in a more general setting. In fact, it implies that for θ of the type we have chosen the relative entropy $H(Q^\theta|P)$ is finite.

$\square$

Lemma 2.3 *Let $\theta = (\theta^S, \theta^E)$, and suppose that θ^S satisfies Assumption 1.1 and $\theta^E \in \mathcal{K}_2$. Then $E^\theta[\log Z_T^\theta | \mathcal{F}_\tau]$ is finite $P-$a.s. for every stopping time $\tau \leq T$.*

Proof By Theorem 3.3 in [Kaz], the process $M_t = -\int_0^t \theta_s dW_s^\theta,\ 0 \leq t \leq T$, is a Q^θ–BMO martingale. Therefore there exists a constant c that does not depend on τ such that

$$E^\theta \left[\frac{1}{2} \int_\tau^T \|\theta_s\|^2 ds \middle| \mathcal{F}_\tau \right] \leq c.$$

The equation

$$-\int_\tau^T \theta_s dW_s - \frac{1}{2}\int_\tau^T |\theta_s|^2 ds = -\int_\tau^T \theta_s dW_s^\theta + \frac{1}{2}\int_\tau^T |\theta_s|^2 ds$$

yields

$$E^\theta[\log Z_T^\theta|\mathcal{F}_\tau] = E^\theta\left[\frac{1}{2}\int_\tau^T \|\theta_s\|^2 ds \middle| \mathcal{F}_\tau\right] < \infty.$$

□

2.2 Maximization for start at stopping time

So far we determined the individual utility maximizing investment strategy of an agent on our market, completed by the insurance asset X^E with parameter θ^E for the market price of external risk fixed, who starts trading at time 0. We now show that he might as well start acting at a stopping time τ that takes its values in $[0,T]$ without having to modify his optimal investment strategy. For this purpose, let us recall the results of [Am] for the maximization of a conditional expectation and apply them to our exponential utility function. Let $\tau \leq T$ denote a stopping time. We want to solve the following conditioned maximization problem:

Problem 2.4 (Individual utility maximization, start at τ)

$$\begin{aligned} J_\tau^a(v_\tau^a, H^a, \theta^S, \theta^E) &= \sup_{\pi\in\mathcal{A}} E[-\exp(-\alpha_a(V_T(v_\tau^a,\pi)+H^a))|\mathcal{F}_\tau] \\ &= \sup_{\pi\in\mathcal{A}} E\left[-\exp\left(-\alpha_a\left(v_\tau^a + \int_\tau^T \pi_s \frac{dX_s}{X_s} + H^a\right)\right)\middle|\mathcal{F}_\tau\right]. \end{aligned}$$

Hereby the initial capital v_τ^a is an $\mathcal{F}_\tau$–measurable random variable, the wealth process of an admissible trading strategy a Q^θ– supermartingale. Extending the arguments made above to reformulate the optimization problem in terms of maximization over a budget set, and in particular using Doob's optional stopping theorem, we find that the problem may be recast in the following way. Define the budget set $\mathcal{B}(\tau, v_\tau^a, H^a, \theta^S, \theta^E)$ using the conditional expectation with respect to $\mathcal{F}_\tau$ by

$$\mathcal{B}(\tau, v_\tau, H^a, \theta^S, \theta^E) := \{D \in L^1(Q^\theta, \mathcal{F}_T) \ : \ E^\theta[D|\mathcal{F}_\tau] \leq v_\tau + E^\theta[H^a|\mathcal{F}_\tau] \ P-a.s.\} \quad (12)$$

(see [Am] Proposition 4.3). Then we have to solve a maximization problem concerning random variables which represent the agents' individual claims:

$$J_\tau^a(v_\tau^a, H^a, \theta^S, \theta^E) = \sup_{D\in\mathcal{B}(\tau,v_\tau^a,H^a,\theta^S,\theta^E)} E[-\exp(-\alpha_a D)|\mathcal{F}_\tau]. \quad (13)$$

The exponential utility function does not satisfy the hypothesis made in [Am]. But it is easy to apply the same method in our case. In fact, again an application of the Fenchel–Legendre transform will yield the result with the usual arguments.

Theorem 2.5 *Let H^a be a bounded $\mathcal{F}_T$–measurable random variable, v^a_τ an $\mathcal{F}_\tau$– measurable random variable. Define*

$$D^{a,\tau}(\theta^S,\theta^E) := D^{a,\tau}(v_0^a, H^a, \theta^S, \theta^E) = -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\Lambda_a Z^\theta_\tau),$$

where Λ_a is an $\mathcal{F}_\tau$–measurable random variable which satisfies

$$-\frac{1}{\alpha_a}\log\Lambda_a = v^a_\tau + E^\theta[H^a|\mathcal{F}_\tau] + \frac{1}{\alpha_a}\log\frac{1}{\alpha_a} + \frac{1}{\alpha_a}E^\theta[\log Z^\theta_T|\mathcal{F}_\tau].$$

Then $D^{a,\tau}(\theta^S,\theta^E)$ is the solution of the utility maximization problem (2.4)for agent $a \in I$.

Proof Our reasoning via Theorem 2.3.2 of [KLS2] this time leads us to the problem of finding an $\mathcal{F}_\tau$–measurable random variable which satisfies

$$-\frac{1}{\alpha_a}\log\Lambda_a = v^a_\tau + E^\theta[H^a|\mathcal{F}_\tau] + \frac{1}{\alpha_a}\log\frac{1}{\alpha_a} + \frac{1}{\alpha_a}E^\theta[\log Z^\theta_T|\mathcal{F}_\tau].$$

This again boils down to a finite relative entropy condition already covered by Lemma 2.3.

□

Let us summarize our findings of this section for ease of later reference by giving an explicit formula for the utility maximizing wealth at time T of agent $a \in I$ if he uses his optimal strategy from a stopping time $\tau \leq T$ on with a Q^θ–integrable $\mathcal{F}_\tau$–measurable initial capital v^a_τ. We recall that the parameter θ determines uniquely the second security X^E on our market which is a possible candidate for making the external risk tradable. The formula we obtain from Theorem 2.5 by employing the explicit structure of the density Z^θ_τ reads

$$\begin{aligned} D^{a,\tau}(\theta^S,\theta^E) &= -\frac{1}{\alpha_a}\log\left(\frac{\Lambda_a}{\alpha_a}\right) + \frac{1}{\alpha_a}\int_\tau^T (\theta^S_t dW^1_t + \theta^E_t dW^2_t) \\ &\quad + \frac{1}{2\alpha_a}\int_\tau^T (|\theta^S_t|^2 + |\theta^E_t|^2)dt. \end{aligned} \tag{14}$$

To emphasize its explicit dependence on the price of external risk, we further write $\pi^a(\theta^E)$ for the utility maximizing trading strategy attaining the claim

$$D^a(\theta^S,\theta^E) - H^a = V_T(v^a_0, \pi^a(\theta^E)) = v^a_0 + \int_0^T \pi^a(\theta^E)_s \frac{dX_s}{X_s}. \tag{15}$$

The optimal trading strategy satisfies the principle of dynamic programming: if at time $t = 0$ an agent a chooses the optimal strategy $\pi^a(\theta^E)$ which provides the wealth $V_\tau(v^a_0, \pi^a(\theta^E))$ at a stopping time τ, he has to follow the same strategy if he starts acting at time τ with initial capital $V_\tau(v^a_0, \pi^a(\theta^E))$.

3 Partial equilibrium

Let us now introduce our concept of partial equilibrium for the market on which the external risk due to the risk process K is traded. Let us briefly recall the model components implemented so far. Every agent $a \in I$ obtains an initial capital v_0^a and at time T a random risky income H^a that, besides the economic development described by the exogenous stock price process X^S, depends on the external risk process K. A second (insurance) security X^E is created to make individual risks immanent in the incomes H^a and caused by K tradable. It depends on the process parameter θ^E which describes a possible price of external risk in X^E. Given such a system of pricing risk every agent trades with X^S and X^E and calculates the trading strategy $\pi^a(\theta^E)$ that maximizes expected exponential utility with individual risk aversion α_a of the sum of his terminal wealth from trading and the income H^a. In order to reach a partial equilibrium, we have to find a market price of external risk process $\theta^{E*} \in \mathcal{K}_2$ for which at any time t a market clearing condition for the second security is satisfied, i.e. $\sum_{a\in I} \pi_{2,t}(\theta^{E*}) = 0$. This equilibrium is called partial since no market clearing for the stock X^S is required.

Definition 3.1 (partial equilibrium) *Let the initial capitals $v_0^a \in \mathbf{R}$, the terminal incomes H^a, $a \in I$, and the stock price process X^S be given. A* partial equilibrium *consists of a market price of external risk process $\theta^{E*} \in \mathcal{K}_2$ for the second security and trading strategies $\pi^a(\theta^{E*})$, $a \in I$, which satisfy the following conditions:*

1. *for any $a \in I$ the trading strategy $\pi^a(\theta^{E*})$ is the solution of the utility maximization problem 2.1 for the stock price process X^S and the price process of the second security associated with market price of risk θ^{E*},*
2. *the second component $\pi_2^a(\theta^{E*})$, $a \in I$, satisfies the* partial market clearing condition
$$\sum_{a\in I} \pi_2^a(\theta^{E*}) = 0 \qquad l \otimes P - a.e.$$

The condition that the market be in partial equilibrium puts a natural constraint on the set of processes of market price of risk for the second security. We shall now investigate the impact of this constraint. It will completely determine the structure of θ^{E*} and therefore also a unique martingale measure Q^{θ^*} obtained via (6) for $\theta^* = (\theta^S, \theta^{E*})$. So we shall have to compute θ^E from the condition that the market be in partial equilibrium with respect to $X^E = X_0^E + \int_0^\cdot X_s^E\, \sigma_s^E\, (dW_s^2 + \theta_s^E ds)$. Recall that Assumption 1.1 guarantees $\theta^S \in \mathcal{H}^\infty(P, \mathbf{R})$. In the following Lemma the overall effect of the partial equilibrium condition emerges. Plainly, if we take the sum of the terminal incomes and terminal wealth obtained by all agents from trading on the security market composed of X^S and X^E, the condition of partial equilibrium just eliminates the contribution of X^E.

Lemma 3.2 *Let $\theta = (\theta^S, \theta^E)$ be such that θ^S satisfies Assumption 1, and $\theta^E \in \mathcal{K}_2$. The market is in partial equilibrium if and only if there exist an $\mathbf{F}$–predictable real* *valued stochastic process ϕ with $E^\theta\left[\left(\int_0^T(\phi_s)^2ds\right)^{\frac{1}{2}}\right] < \infty$ such that the optimal claims $(D^a(\theta^S,\theta^E))_{a\in I}$ and incomes $(H^a)_{a\in I}$ satisfy the equation*

$$\sum_{a\in I}(D^a(\theta^S,\theta^E) - H^a) = c_0 + \int_0^T \phi_s(dW_s^1 + \theta_s^S ds) \tag{16}$$

with some constant $c_0 \in \mathbf{R}$. Hence $\pi = (\pi_1, 0)$ with $\pi_1 = \phi(\sigma^S)^{-1} = \sum_{a\in I}\pi_1^a$, possesses the properties of an admissible trading strategy.

Proof First we apply the representation property (8) to the terminal wealth $D^a(\theta^S,\theta^E) - H^a$ of each individual agent $a \in I$ with initial capital v_0^a, then sum over all $a \in I$. Using linearity of the stochastic integral and recalling (7) we thus obtain

$$\begin{aligned}
&\sum_{a\in I}(D^a(\theta^S,\theta^E) - H^a) \\
&= \sum_{a\in I} v_0^a + \int_0^T (\sum_{a\in I}\pi_{1,t}^a)\frac{dX_t^S}{X_t^S} + \int_0^T (\sum_{a\in I}\pi_{2,t}^a)\frac{dX_t^E}{X_t^E} \\
&= \sum_{a\in I} v_0^a + \int_0^T (\sum_{a\in I}\pi_{1,t}^a)\,\sigma_t^S(dW_t^1 + \theta_t^S dt) \\
&\quad + \int_0^T (\sum_{a\in I}\pi_{2,t}^a)\,\sigma_t^E(dW_t^2 + \theta_t^E dt).
\end{aligned} \tag{17}$$

To prove the 'only if' part, write now $\pi_i = \sum_{a\in I}\pi_i^a$, $i = 1,2$. Since the market is in partial equilibrium, we have $\pi_2 = 0$. Hence the desired equation (16) follows.

For the 'if' part, suppose that $\sum_{a\in I}(D^a(\theta^S,\theta^E) - H^a)$ can be written as in (16). By comparison with (17) and uniqueness of integrands in stochastic integral representations we obtain $\pi_1 = \frac{\phi}{\sigma^S}$ and $\pi_2 = 0$. This establishes the equivalence. Finally, $\pi = (\pi_1, 0)$ is admissible, because $\sum_{a\in I}(D^a(\theta^S,\theta^E) - H^a) \in L^1(Q^\theta)$, and the process $\int_0^\cdot \pi_{1,t}dX_t^S$ is even a Q^θ–martingale.

□

3.1 Existence of partial equilibrium

We now come to the main goal of this section, the construction of θ^E for which the partial equilibrium constraint is satisfied. At the same time, this will justify the existence of a partial equilibrium. We use the characterization of the utility maximizing payoffs in a partial equilibrium described in Lemma 3.2 and the explicit formula (14). This will enable us to describe θ^{E*} and ϕ (or π) in terms of the solution of a BSDE. To abbreviate, we write

$$\bar{\alpha} = (\sum_{a\in I}\frac{1}{\alpha_a})^{-1}, \quad \bar{H} = \sum_{a\in I}H^a + \frac{1}{2\bar{\alpha}}\int_0^T |\theta_s^S|^2 ds. \tag{18}$$

We combine the two alternative descriptions of $\sum_{a\in I}(D^a(\theta^S,\theta^{E*}) - H^a)$ provided by Lemma 3.2 and the equation

$$\begin{aligned}
&\sum_{a\in I}(D^a(\theta^S,\theta^{E*}) - H^a) \qquad (19)\\
&= c_1 + \frac{1}{\bar\alpha}\int_0^T (\theta_t^S\, dW_t^1 + \theta_t^{E*}\, dW_t^2) + \frac{1}{2\bar\alpha}\int_0^T (|\theta_t^S|^2 + |\theta_t^{E*}|^2)dt - \sum_{a\in I} H^a
\end{aligned}$$

which follows from (14) with a constant c_1 not specified further at this point, to obtain a condition determining θ^{E*} in the form of a BSDE. To keep to the habits of the literature on BSDE, set

$$\begin{aligned}
z^S &= \theta^S - \bar\alpha\phi,\\
z^E &= \theta^{E*}.
\end{aligned}$$

In this notation the comparison of (16) and (19) yields the equation

$$h_0 = \bar\alpha\bar H - \int_0^T (z_t^S dW_t^1 + z_t^E dW_t^2) - \int_0^T \frac{1}{2}|z_t^E|^2\, dt - \int_0^T \theta_t^S z_t^S\, dt. \qquad (20)$$

Due to Assumption 1.1, $\bar H$ is bounded. By extending (20) from time 0 to any time $t \in [0,T]$ we obtain a BSDE whose solution uniquely determines $z^E = \theta^{E*}$. It defines backward in time a predictable stochastic process $(h_t)_{t\in[0,T]} \in \mathcal{H}^\infty(\mathbf{R}, P)$ with terminal value $h_T = \bar\alpha\bar H$ and an integrand $(z_t = (z_t^S, z_t^E))_{t\in[0,T]} \in \mathcal{H}^2(\mathbf{R}^2, P)$. The following Theorem provides an equilibrium solution by setting $\theta^{E*} := z^E$ which is obtained from known results on non-linear BSDE.

Theorem 3.3 *The backwards stochastic differential equation (BSDE)*

$$h_t = \bar\alpha\bar H - \int_t^T (z_s^S dW_s^1 + z_s^E dW_s^2) - \int_t^T \theta_s^S z_s^S ds - \int_t^T \frac{1}{2}|z_s^E|^2 ds, \qquad (21)$$

$t \in [0,T]$, *possesses a unique solution given by the triple of processes* $(h,(z^S,z^E)) \in \mathcal{H}^\infty(P,\mathbf{R}) \times \mathcal{H}^2(P,\mathbf{R}^2)$. *The choice* $\theta^{E*} := z^E$ *provides a partial equilibrium for the market.*

Proof $\bar H$ is $\mathcal{F}_T$–measurable and bounded. The process θ^S is $\mathbf{F}$–predictable and uniformly bounded in (ω,t). By Theorem 2.3 and Theorem 2.6 in [Kob], equation (21) has a unique solution $(h,(z^S,z^E)) \in \mathcal{H}^\infty(P,\mathbf{R}) \times \mathcal{H}^2(P,\mathbf{R}^2)$. Let then $\theta^{E*} := z^E$ and $\phi := \frac{1}{\bar\alpha}(\theta^S - z^S)$. Then, thanks to Lemma 3.2 we get a partial equilibrium, provided we can prove that $z^E \in \mathcal{K}_2$. This is done in Lemma 3.4 below. Given θ^{E*}, for the coefficients b^{E*} and σ^{E*} we are free to choose for example

$$b^{E*} = \theta^{E*}, \qquad \sigma^{E*} = 1.$$

□

Lemma 3.4 *Let z^E be the third component of the solution $(h,(z^S,z^E))$ of (21). Then the process $M=\int_0^\cdot z^E_s dW^2_s$ is a P–BMO martingale.*

Proof Without loss of generality, we may suppose $K=\bar{\alpha}\bar{H}$ nonnegative. To see this, recall that $\bar{\alpha}\bar{H}$ is bounded from below by a constant S. We may then solve the BSDE (21) for $K=\bar{\alpha}\bar{H}-S$ instead. By uniqueness its solution $(k,(y_1,y_2))$ satisfies $k=h-S, y_1=z^S, y_2=z^E$. If $K\geq 0$, the comparison theorem (Theorem 2.6 [Kob]) gives $h\geq 0$. For every stopping time $\tau\leq T$, Itô's formula yields

$$\begin{aligned} & E\left[K^2-h_\tau^2-\int_\tau^T (2h_s\theta_s^S z_s^S+|z_s^S|^2)ds\,\middle|\,\mathcal{F}_\tau\right] \\ = & E\left[\int_\tau^T (h_s+1)|z_s^E|^2 ds\,\middle|\,\mathcal{F}_\tau\right]\geq E\left[\int_\tau^T |z_s^E|^2 ds\,\middle|\,\mathcal{F}_\tau\right]. \end{aligned}$$

To find also an upper bound for the left hand side in the inequality above we note

$$-2h_s\theta_s^S z_s^S-|z_s^S|^2=|\theta_s^S|^2h_s^2-(\theta_s^S h_s+z_s^S)^2.$$

Let S_1 denote an upper bound for K^2 and S_2 an upper bound for $|\theta_s^S|^2h_s^2$. Then we get for every stopping time $\tau\leq T$

$$\begin{aligned} S_1+TS_2 &\geq E\left[\int_\tau^T |z_s^E|^2 ds\,\middle|\,\mathcal{F}_\tau\right] \\ &= E\left[\langle M\rangle_T-\langle M\rangle_\tau\,\middle|\,\mathcal{F}_\tau\right]. \end{aligned}$$

Therefore M is a P–BMO martingale.

□

3.2 Uniqueness of partial equilibrium

In the following subsection we shall show that the choice $\theta^{E*}=z^E$ made above provides the unique equilibrium price of external under the assumptions valid for the coefficient processes.

Theorem 3.5 *Suppose $\theta^{E*}=b^{E*}/\sigma^{E*}$ is such that the market is in a partial equilibrium. Then $z^E=\theta^{E*}$ is the third component of the unique solution process $(h,(z^S,z^E))$ of (21).*

Proof We first apply Girsanov's Theorem to eliminate the known drift θ^S from our considerations. More formally, consider the probability measure $\tilde{Q}$ given by the density

$$\frac{d\tilde{Q}}{dP}=\mathcal{E}\left(-\int_0^T(\theta_t^S,0)dW_t\right).$$

Let $\tilde{W}=W+\int_0^\cdot(\theta_s^S,0)ds$ be the corresponding Brownian motion under $\tilde{Q}$.

Now define $z^S = \theta^S - \bar{\alpha}\phi$, $z^E = \theta^{E*}$ and $z_t = (z_t^S, z_t^E)^{tr}$. Since z^E guarantees that the market is in partial equilibrium, as for (20) we deduce with a constant c

$$\begin{aligned} c &= \bar{\alpha}\bar{H} - \int_0^T (z_t^S dW_t^1 + z_t^E dW_t^2) - \int_0^T \frac{1}{2}|z_t^E|^2\, dt - \int_0^T \theta_t^S z_t^S\, dt \qquad (22) \\ &= \bar{\alpha}\bar{H} - \int_0^T z_t d\tilde{W}_t - \int_0^T \frac{1}{2}|z_t^E|^2\, dt. \end{aligned}$$

Hence we may further define the process h by

$$h_t = c + \int_0^t z_s d\tilde{W}_s + \frac{1}{2}\int_0^t (z_s^E)^2 ds,$$

with the alternate description

$$h_t = \bar{\alpha}\bar{H} - \int_t^T z_s d\tilde{W}_s - \int_t^T \frac{1}{2}|z_s^E|^2 ds, \quad t \in [0,T]. \qquad (23)$$

This yields that $(h, (z^S, z^E))$ solves (21). It remains to verify according to Theorem 2.6 in [Kob] that

$$\begin{aligned} &(z^S, z^E) \in \mathcal{H}^2(P, \mathbf{R}^2), \\ &h \text{ is uniformly bounded.} \end{aligned}$$

Let us first argue for the square integrability of (z^S, z^E). By the definition of the partial equilibrium, we have $\theta^E \in \mathcal{H}^2(P, \mathbf{R})$. θ^S being bounded, it remains to argue for P-square-integrability of ϕ, where ϕ is given by (16). By Burkholder-Davis-Gundy's inequality, we have $\sum_{a\in I}(D^a(\theta^S, \theta^E) - H^a) \in L^p(\tilde{Q})$ for $p \geq 1$, and this random variable can be represented as a stochastic integral with the integrand $(\phi, 0)$ with respect to the Brownian motion $\tilde{W}$. Hence,

$$E^{\tilde{Q}}([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) < \infty,$$

for $p \geq 1$. Therefore, due to Hölder's inequality and

$$E^P([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) = E^{\tilde{Q}}([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}} \mathcal{E}(\int_0^T (\theta_t^S, 0) d\tilde{W}_s))$$

we also obtain

$$E^P([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) < \infty$$

for all $p \geq 1$.

To prove the boundedness of h, we perform still another equivalent change of measure. Let $\hat{Q}$ be given by

$$\frac{d\hat{Q}}{dP} = \mathcal{E}(-\int_0^T (\theta_t^S, \frac{1}{2} z_t^E) dW_t).$$

Then by virtue of (23) we get

$$h_t = E^{\hat{Q}}[\bar{\alpha}\bar{H}|\mathcal{F}_t], \quad t \in [0,T].$$

Therefore h has a uniformly bounded version with the same bounds as $\bar{\alpha}\bar{H}$.

□

We conclude this section by showing that the unique equilibrium constructed persists if the individual utility maximization problems of the agents on the market start at some stopping time τ.

Remark 3.6 *The market price of risk θ^{E*} that attains a partial equilibrium satisfies a dynamic programming principle. Indeed, let θ^{E*} be the unique market price of risk process in $\mathcal{K}_2$ calculated for the individual utility maximization starting at time $t = 0$. Let $\tau \leq T$ be a stopping time and let the agents solve the conditioned maximization problem 2.4 beginning at time τ with terminal incomes H^a. Then the partial equilibrium is given by θ^{E*} as well.*

For the construction of a partial equilibrium for trading after τ we proceed in the same way as in the case of the maximization of a conditioned expected utility. The definition of a partial equilibrium remains as in Definition 3.1. The starting point is Lemma 3.2 adapted to the sigma–algebra $\mathcal{F}_\tau$, where we have to replace the constant c_0 by an $\mathcal{F}_\tau$–measurable bounded random variable c_τ. Comparing the explicit solution of the utility maximization with respect to a candidate for an equilibrium market price of risk process θ^{E*} to (16) yields the following BSDE with $z = (\tilde{z}^S, \tilde{z}^E)$

$$\tilde{h}_t = \bar{\alpha} \sum_{a \in \mathcal{A}} H^a - \int_t^T \tilde{z}_s dW_s - \int_t^T \left(\frac{1}{2} |\tilde{z}_s^E|^2 + \theta^S \tilde{z}_s^S - \frac{1}{2} |\theta_s^S|^2 \right) ds, \quad t \in [\tau, T].$$

By uniqueness of the solution of the BSDE, we derive $\tilde{h}_t = h_t + \int_0^t \frac{1}{2} |\theta_s^S|^2 ds$ and for the integrands $(\tilde{z}^E, \tilde{z}^S) = (z^E, z^S)$. As for the utility maximization beginning at $t = 0$ we obtain $\theta^{E*} = z^E$ and $\phi = \frac{1}{\bar{\alpha}}(\theta^S - z^S)$. The market price of risk process $\theta^{E*} \in \mathcal{K}_2$ that attains the partial equilibrium is unique. The proof of Theorem 3.5 remains valid if we replace the constant c in (22) with an $\mathcal{F}_\tau$– measurable bounded random variable.

4 Partial equilibrium without a second security

In this section, we shall describe an alternative approach to the problem of transferring external risks by trading on a financial market in partial equilibrium. This approach is conceptually more flexible and therefore better appropriate for dealing with risk exposures too complicated to be tradable by just one security. The ingredients of the model are basically the same.

There is a stock market with a stock evolving according to an exogenous price process X^S. As in section 2, we consider finitely many agents $a \in I$ each one of which is endowed with an initial capital v_0^a and a random income H^a payed out at the terminal time T. H^a depends on the economic development described by X^S and a process K representing external risk which cannot be hedged by trading on the stock market. In this section we do not construct a second security to be traded together with X^S.

Instead, the agents have the possibility to sign mutual or multilateral contracts in order to exchange *random payoffs* in addition to trading with the stock.

Let us first explain what corresponds to *market completion* in this version of the model. The agents' random payoffs are priced using one and the same pricing rule for the entire market. The value of a payoff that is replicable by a trading strategy must be equal to the initial capital of the trader. Therefore, a pricing rule that is consistent with the stock price is linear on the replicable payoffs. We only consider pricing rules which are linear on all payoffs. It is well known that pricing rules that are continuous linear functionals on an $L^p(P)$-space for some $p > 1$ and preserve constants can be described as expectations of a probability measure absolutely continuous with respect to P. Under the additional assumption that a nontrivial positive payoff has a positive price, these probability measures turn out to be equivalent to P. A pricing rule meeting all these claims and being consistent with the stock price is therefore given by the expectation under a probability measure equivalent to P for which X^S is a martingale. We call those measures *pricing measures*.

Given a particular pricing measure Q, every agent possesses a budget set which must contain those random payoffs that are cheaper than the sum of his initial capital and the value of his income H^a. The preferences of an agent a are described by the expected exponential utility with individual risk aversion α_a. Now every agent maximizes his utility by choosing the best priced payoff in his budget set under Q. He then has to replicate the difference between this payoff and his income H^a by trading with the stock, which is possible since the stock price process is a martingale under Q, and signing contracts with other agents.

And here is how we interpret *partial equilibrium* in this setting. Fix again a pricing measure Q for a moment. The random claim of each agent a may be decomposed into a part which is hedgeable under Q purely with X^S, and an additional part C^a which depends on Q and describes the remaining compound risk of his contracts with other agents. So we have to look for an *equilibrium pricing measure* Q^* for which the total compound risk $\sum_{a\in I} C^a$ vanishes. In other terms, the difference of offers and demands of payoffs by the different agents creates a claim they are able to hedge on the financial market alone.

We use a version of the explicit formula (14) for the utility maximizing payoff and the partial market clearing condition to characterize the density of the pricing measure that attains the equilibrium in terms of the solution of a BSDE as before.

4.1 Alternative completion

This time we work on a d–dimensional model with a Brownian motion $W = (W_1, \dots, W_d)$. The P–completion of the filtration generated by W is denoted by $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$. As in (1) the stock price process X is given by the stochastic equation

$$X_t^S = X_0^S + \int_0^t X_s^S (b_s^S ds + \sigma_s^S dW_s^1), \quad t \in [0, T]. \tag{24}$$

The basic facts about our model remain unchanged with respect to the previous sections. The coefficients b^S and σ^S satisfy Assumption 1.1 and therefore $\theta^S := b^S/\sigma^S$

is $\mathbf{F}$–predictable and uniformly bounded. The process K that describes the external risk is $\mathbf{F}$–adapted. For $a \in I$ the income H^a that agent a receives at time T is again a real-valued bounded $\mathcal{F}_T$–measurable random variable of the form

$$H^a = g^a(X^S, K).$$

Every agent a is endowed at time $t = 0$ with an initial capital $v_0^a \geq 0$, and maximizes his expected utility with respect to the exponential utility function

$$U^a(x) = -\exp(-\alpha_a x), \quad x \in \mathbf{R},$$

with an individual risk aversion coefficient $\alpha_a > 0$.

According to the introductory remarks we next specify the system of prices admitted for pricing the claims of agents on our market. We aim at considering pricing measures which do not change prices for X^S. Hence we let $\mathbf{P}_e$ be the collection of all probability measures Q on $\mathcal{F}_T$ which are equivalent to P and such that X^S is a Q–martingale.

The price of a claim B under $Q \in \mathbf{P}_e$ is described by the expectation

$$E^Q[B] \tag{25}$$

which makes sense for all contingent claims such that this expectation is well defined, e.g. for B bounded from below. The set of equivalent martingale measures $\mathbf{P}_e$ parameterizes all linear pricing rules that are continuous in an $L^p(P)$– space for $p > 1$, strictly positive on $L^0_+(P)\backslash\{0\}$ and consistent with the stock price process X^S. These pricing systems do not allow arbitrage.

$\mathbf{P}_e$ can be described and thus parameterized explicitly. It consists of all probability measures Q^θ possessing density processes with respect to P of the following form

$$\left.\frac{dQ^\theta}{dP}\right|_{\mathcal{F}_t} = Z_t^\theta = \mathcal{E}\left(-\int (\theta_s^S, \theta_s^E) dW_s\right)_t, \quad t \in [0, T], \tag{26}$$

with a predictable $\mathbf{R}^{d-1}$–valued process θ^E such that the stochastic exponential is a uniformly integrable martingale. We denote $\theta = (\theta^S, \theta^E)$. The process θ^E plays the same part as in section 3. Using this parametrized set, the strategies agents are allowed to use can be formulated in the following way.

Definition 4.1 (admissible trading strategy, wealth process) *An admissible trading strategy with initial capital $v_0 \geq 0$ is a stochastic process π with $\int_0^T |\sigma_s^S \pi_s|^2\, ds < \infty$ P–a.s. and such that there exists a probability measure $Q^\theta \in \mathbf{P}_e$ such that the wealth process*

$$V_t(v_0, \pi) = v_0 + \int_0^t \pi_s \frac{dX_s^S}{X_s^S}, \qquad t \in [0, T],$$

is a Q^θ–supermartingale.

The set of admissible trading strategies is free of arbitrage. A strategy π with a wealth process $V(v_0, \pi)$ that is bounded from below is admissible.

4.2 Utility maximization

For the purpose of utility maximization with respect to our exponential utility functions the set $\mathbf{P}_e$ has to be further restricted to the set $\mathbf{P}_f$ of equivalent martingale measures with finite relative entropy with respect to P (see section 2). Let $Q^\theta \in \mathbf{P}_f$ for $\theta = (\theta^S, \theta^E)$ be given. The condition under which agents maximize their expected utility is given by a *budget constraint*. An individual agent a can choose among all claims that are not more expensive than the sum of his initial capital v_0^a and the price of his income $E^\theta(H^a) = E^{Q^\theta}(H^a)$. The set of these claims is called the budget set for agent a, formally given by

$$\mathcal{B}^a := \mathcal{B}(v_0^a, H^a, Q^\theta) = \{D \in L^1(Q^\theta, \mathcal{F}_T) : E^\theta[D] \leq v_0^a + E^\theta[H^a]\}.$$

Every agent a chooses in his budget set the claim $D^a(Q^\theta)$ that maximizes his expected utility, i.e. the solution of the following maximization problem

$$J^a(v_0^a, H^a, Q^\theta) = \sup_{D \in \mathcal{B}(v_0^a, H^a, Q^\theta)} E[-\exp(-\alpha_a D)]. \tag{27}$$

According to the well known theory of utility maximization via Fenchel–Legendre transforms, the solution is given by the following Theorem. Here we put $I^a(y) = ((U^a)')^{-1}(y) = -\frac{1}{\alpha_a}\log\frac{y}{\alpha_a}$, for $Q^\theta \in \mathbf{P}_f$. Note that taking Q^θ from this set replaces an appeal to Lemma 2.3 in the proof.

Theorem 4.2 *Let H^a be a bounded $\mathbf{F}_T$–measurable random variable, $v_0^a \geq 0$. Define*

$$D^a(Q^\theta) = I(\lambda_a Z_T^\theta) = -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta), \tag{28}$$

where λ_a is the unique real number such that

$$E^\theta[I^a(\lambda_a Z_T^\theta)] = v_0^a + E^\theta[H^a].$$

Then $D^a(Q^\theta)$ is the solution of the utility maximization problem (27)for agent $a \in I$.

4.3 Partial equilibrium

Let us now describe more formally what we mean by a partial equilibrium. We want to construct a stochastic process θ^{E*} and with $\theta^* = (\theta^S, \theta^{E*})$ via (26) a measure $Q^* = Q^{\theta*} \in \mathbf{P}_f$ under which the overall difference between demands and offers of agents' claims is replicable on the financial market, i.e. can be hedged with the security X^S. In different terms, we look for a price measure Q^* such that $\sum_{a\in I}(D^a(Q^*) - H^a)$ can be represented as a stochastic integral with respect to the stock price process X with an integrand given by an admissible trading strategy. Under Q^θ, agent a knows the claim $D^a(Q^\theta)$ which maximizes his expected utility. He covers the difference $D^a(Q^\theta) - H^a$ between his preferred payoff and his income by two components: the terminal wealth of a trading strategy $\pi^a(Q^\theta)$, and the payoff $C^a(Q^\theta)$ from the mutual contracts with the other participants in the market. Formally,

$$D^a(Q^\theta) - H^a = C^a(Q^\theta) + v_0^a + \int_0^T \pi^a(Q^\theta)_s \frac{dX_s^S}{X_s^S}.$$

We now define the partial equilibrium measure Q^* by claiming that

$$\sum_{a\in I} C^a(Q^*) = 0.$$

Definition 4.3 (partial equilibrium) *Let $(H^a)_{a\in I}$ be a family of bounded $\mathcal{F}_T$–measurable incomes, $(v_0^a)_{a\in I}$ a family of initial capitals of the agents, X^S the exogenous stock price process according to (24), $(U^a)_{a\in I}$ a family of exponential utility functions with risk aversion coefficients $(\alpha_a)_{a\in I}$, and $(D^a(Q^\theta))_{a\in I}$ the family of utility maximizing claims according to (28) for $Q \in \mathbf{P}_f$. A probability measure $Q^* \in \mathbf{P}_f$ attains the* partial equilibrium *if there exists an admissible trading strategy π^* such that we have*

$$\sum_{a\in I}(D^a(Q^*) - H^a) = \sum_{a\in I} v_0^a + \int_0^T \pi_s^* \frac{dX_s^S}{X_s^S}.$$

In view of the preceding remarks, to obtain the admissible trading strategy π^* of Definition 4.3 we have to sum all the individual strategies $\pi^a(Q^*)$ of agents a over $a \in I$. Given the equilibrium measure, the existence of π^* is equivalent to the existence of an $\mathbf{F}$–predictable real valued stochastic process ϕ^* satisfying

$$\sum_{a\in I}(D^a(Q^*) - H^a) = \sum_{a\in I} v_0^a + \int_0^T \phi_t^*(dW_t^1 + \theta_t^S dt).$$

The process ϕ^* and the admissible trading strategy π^* are related by the equation

$$\pi^* = \frac{\phi^*}{\sigma^S}.$$

To construct Q^*, we just have to find an appropriate process θ^{E*} appearing in the exponential of an equivalent measure change and take $Q^* = Q^{(\theta^S,\theta^{E*})}$. But this just means that we can proceed as in section 3 and use the technology of BSDE. The process θ^{E*} will just be the higher dimensional version of the process θ^{E*} constructed there. Since we are in a d–dimensional model here, we shall give a few details of the analogous construction. Let

$$\bar{H} = \sum_{a\in I} H^a + \frac{1}{2\bar{\alpha}} \int_0^T |\theta_t^S|^2 dt,$$

$z_1 = \theta_t^S - \bar{\alpha}\phi_t^*$, $z_i = \theta_{i-1}^E$, $i = 2, \dots d$. We obtain the following BSDE

$$h_t = \bar{\alpha}\bar{H} - \int_t^T (z_{1,s}, \dots, z_{d,s}) dW_s - \int_t^T \theta_s^S z_{1,s} ds - \frac{1}{2}\sum_{i=2}^d \int_t^T (z_{i,s})^2 ds,$$

$t \in [0,T]$. The process θ^S is uniformly bounded by Assumption 1.1 and $\bar{H}$ is also bounded. In this setting the following existence result for a partial equilibrium holds.

Theorem 4.4 *The backwards stochastic differential equation (BSDE)*

$$h_t = \bar{\alpha}\bar{H} - \int_t^T (z_{1,s}, \ldots, z_{d,s}) dW_s - \int_t^T \theta_s^S z_{1,s} ds - \frac{1}{2} \sum_{i=2}^d \int_t^T (z_{i,s})^2 ds, \tag{29}$$

$t \in [0,T]$, possesses a unique solution given by the triple of processes $(h, (z^S, z^E)) \in \mathcal{H}^\infty(P, \mathbf{R}) \times \mathcal{H}^2(P, \mathbf{R}^d)$. The choice $\theta^{E*} = (z_2, \ldots, z_d)$ and Q^* defined via (26) with (θ^S, θ^{E*}) gives a pricing measure for which a partial equilibrium is attained.

Proof Due to Theorem 2.3 and Theorem 2.6 in [Kob], (29) possesses a unique solution $(h, z) \in \mathcal{H}^\infty(P, \mathbf{R}) \times \mathcal{H}^2(P, \mathbf{R}^d)$. Now set

$$\theta^{E*} = (z_2, \cdots, z_d). \tag{30}$$

As in Lemma 3.4 it follows that $\int_0^\cdot (\theta_s^S, \theta_s^{E*}) dW_s$ is a P–BMO martingale. The stochastic exponential $\mathcal{E}(-\int (\theta_s^S, \theta_s^{E*}) dW_s)$ is a uniformly integrable martingale and the Radon–Nikodym density of a probability measure $Q^* \in \mathbf{P}_e$ with respect to P. As in Lemma 2.3, we get $H(Q^*|P) < \infty$ and by (14) and (11) the maximal utility for every agent is finite. By virtue of $\phi^* = \frac{1}{\bar{\alpha}}(\theta^S - z_1)$, $(\int_0^t \phi_s^* (dW_s^1 + \theta_s^S ds))_{t \in [0,T]})$ is a Q^*–martingale. Hence, Q^* defines via (26) a pricing measure that attains the partial equilibrium.

□

For the corresponding uniqueness result, we need the technical condition that the stochastic integral process associated with (θ^S, θ^{E*}) belongs to BMO.

Theorem 4.5 *Let $Q^{(\theta^S, \theta^{E*})} \in \mathbf{P}_f$ attain the partial equilibrium and suppose that $(\int_0^t (\theta_s^S, \theta_s^{E*}) dW_s)_{t \in [0,T]}$ is a P–BMO martingale. Then we have $\theta^{E*} = (z_2, \ldots, z_d)$ and $\phi^* = \frac{1}{\bar{\alpha}}(\theta^S - z_1)$ where $z = (z_1, \ldots, z_d)$ is given by the solution of (29).*

Proof The proof of this statement is quite similar to the one of Theorem 3.5.

□

We conclude this section by noting that as in section 3, θ^{E*} satisfies a dynamic programming principle.

Remark 4.6 *Let the probability measure Q^* be given through (30) and (26). If the agents solve the conditioned optimization problem 2.4 for a stopping time $\tau \leq T$ with the same incomes (H^a), then Q^* attains also a partial equilibrium.*

The arguments needed to prove this are as for Remark 3.6.

Remark 4.7 *Our simple model clarifies some basic questions. Of course, not only the simulation results in [CIM] raise many more interesting questions than it can answer. Among them are the following. Why does the insurance market only admit a finite number of agents? How does the overall wealth increase depend on the composition of the market? What is the difference between the second security and a re-insurance* *treaty? How can big agents- such as re-insurers - with influence on the price dynamics be included into the model? Under which conditions can the market be liquid? In writing this paper we were aware of these and other questions which our model is unable to tackle at the moment. Forthcoming papers will deal with continuing research on securitization, in particular in the context of climate risk. For example, the model will be refined by introducing big agents on the market, and the financial products available will include for example cat bond type objects.*

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5 Appendix: BMO martingales

Here we recall and collect a few well known facts from the theory of martingales of *bounded mean oscillation*, briefly called BMO–martingales. We follow the exposition in [Kaz]. The statements will be made for infinite time horizon. In the text they will be applied to the simpler framework of finite horizon, replacing ∞ with T.

Definition 5.1 *Let $M = (M_t)_{t\geq 0}$ be a uniformly integrable martingale with respect to a probability measure P and a complete, right continuous filtration $\mathbf{F}$ satisfying $M_0 = 0$. For $1 \leq p < \infty$ set*

$$\|M\|_{BMO_p} := \sup_{\tau\ \mathbf{F}-\textit{stopping time}} E[|M_\infty - M_\tau|^p|\mathcal{F}_\tau]^{1/p}. \tag{31}$$

The normed linear space $\{M : \|M\|_{BMO_p} < \infty\}$ with norm $\|M\|_{BMO_p}$ (taken with respect to P) is denoted by BMO_p (Kazamaki [Kaz], p. 25).

By Corollary 2.1 in [Kaz], p. 28, we have for all $1 \leq p < \infty$

$$M \in \mathrm{BMO}_1 \quad \text{iff} \quad M \in \mathrm{BMO}_p.$$

$BMO(P)$ denotes all uniformly integrable P–martingales such that $\|M\|_{\mathrm{BMO}_1} < \infty$. The norm in $BMO_2(P)$ can be alternatively expressed as

$$\|M\|_{\mathrm{BMO}_2} = \sup_{\tau\ \mathbf{F}-\text{stopping time}} E[\langle M\rangle_\infty - \langle M\rangle_\tau|\mathcal{F}_\tau]^{1/2}. \tag{32}$$

The combined inequalities of Doob and Burkholder–Davis–Gundy read for $p > 1$

$$(\frac{p}{p-1})^p E[|M_\infty|^p] \geq E[\sup_{0\leq t\leq\infty} |M_t|^p] \geq c_p\, E[\langle M\rangle_\infty^{p/2}] \tag{33}$$

with a universal positive constant c_p. Therefore for any BMO–martingale M we obtain $\langle M\rangle_t \in L^p(P)$ for all $p > 1, t \in [0, \infty]$.

BMO-martingales possess the convenient property of generating uniformly integrable exponentials according to the following Theorem.

Theorem 5.2 *(Theorem 2.3 [Kaz]) If $M \in BMO$, then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

According to the following Theorem, the BMO property is preserved by equivalent changes of measure. In fact, let $M \in BMO(P)$ and $\hat{P}$ given by the measure change $d\hat{P} = \mathcal{E}(M)_\infty dP$. Define $\phi : X \mapsto \hat{X} = \langle X, M \rangle - X$.

Theorem 5.3 *(Theorem 3.6 [Kaz]) If $M \in BMO(P)$, then $\phi : X \mapsto \hat{X}$ is an isomorphism of $BMO(P)$ onto $BMO(\hat{P})$.*

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