The cohomology of stochastic and random differential equations, and local linearization of stochastic flows

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Abstract

Random dynamical systems can be generated by stochastic differential equations (sde) on the one side, and by random differential equations (rde), i.e. randomly parametrized ordinary differential equations on the other side. Due to conflicting concepts in stochastic calculus and ergodic theory, asymptotic problems for systems associated with sde are harder to treat. We show that both objects are basically identical, modulo a stationary coordinate change (cohomology) on the state space. This observation opens completely new opportunities for the treatment of asymptotic problems for systems related to sde: just study them for the conjugate rde, which is often possible by simple path-by-path classical arguments. This is exemplified for the problem of local linearization of random dynamical systems, the classical analogue of which leads to the Hartman-Grobman theorem.

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Introduction

The concept of random dynamical system comprises a big variety of different mathematical objects. As their deterministic counterparts, which describe products of maps, or flows of difference or differential equations, they may have a variety of different generators. As opposed to the deterministic setting, however, besides the dynamics on the state space induced by the recursive equations, difference or differential equations, there is a stationary dynamics on the random basis in the background induced by a basis flow, i.e. a (semi)group of measure preserving transformations which interacts with the foreground random motion to produce the mathematical object of cocycles on skew products. Arnold [1] treats this subject along with its generators in depth and detail.

Of particular importance for the treatment of their asymptotic properties such as stability, Lyapunov exponents, random attractors, invariant manifolds, bifurcations, is the variant of ergodic theory suitable for the flow level, namely multiplicative ergodic theory, based on the powerful theorem due to Oseledets [15]. It provides the stochastic analogue of eigenvalues and eigenspaces of a matrix inducing a simple autonomous differential equation. Lyapunov exponents and the Oseledets spectrum play analogous roles for the asymptotic exponential growth of trajectories of linear random cocycles. The particular values asymptotic exponential growth rates can take are described by the Lyapunov numbers, realized by those trajectories whose initial vector belongs to the corresponding Oseledets spaces. These spaces can be described as intersections of forward and backward random flags, and therefore are created by a mixture of $\alpha-$ and $\omega-$limits - a quite common fact in the context of ergodic theory. If the particular cocycle to be treated by these methods of ergodic theory is generated by stochastic differential equations, however, also methods of stochastic analysis, in particular Itô's calculus, enter the scene. And the strict causality behind the stochastic integral notion of Itô is bound to conflict with the causality breaking notions containing time limits at both $\pm\infty$.

For this reason in Arnold's [1] book, there is a clear-cut boundary between the asymptotic treatment of cocycles generated by random matrices or their continuous time counterparts, random differential equations, on the one hand, and stochastic differential equations on the other hand. The border line is marked by the limits of available methods, and the indicated conflict between ergodic theory and stochastic analysis leaves much more question marks and open problems on the side of stochastic differential equations, while on the side of random differential equations, often due to pathwise classical arguments solutions are more readily available.

The bottom line of what we propose to show in this paper is that both classes of cocycles, those generated by random differential equations, and by stochastic differential equations, are basically the same objects. In fact, we shall prove that under only very mild regularity assumptions concerning the vector fields involved there always exist *cohomologies*, i.e. stationary coordinate changes by means of which flows of stochastic differential equations may be viewed as ordinary differential equations with a random parameter. This may remind the reader of the result by Doss and Sussman expressing solutions of sde as solutions of ordinary non-autonomous equations along a deterministic flow. In fact, the idea of a flow decomposition which is behind this and other approaches (see Bismut, Michel [5], [6]) just has to be refined by the requirement that the flow component related to the diffusive part of an sde be stationary with respect to the basis flow. This is achieved by writing its diffusive part in an alternative moving average type representation of the diffusion vector fields with respect to a Wiener process. It exactly parallels the derivation of the stationary Ornstein-Uhlenbeck process as a moving average of a constant vector field with respect to the ordinary Brownian motion.

This surprising result is shown to circumvent elegantly the trouble caused by the conflict of ergodic theory and stochastic analysis: the non-classical fluctuation of the driving noises is absorbed into a stationary fluctuation in the base flow, and thus becomes tractable for ergodic theory. As a consequence, we show that formerly inaccessible asymptotic problems for sde provide solutions via a passage to associated rde for which pathwise classical arguments have already provided solutions of the analogous problems. We consider the local linearization problem for cocycles generated by stochastic differential equations, with the famous Hartman-Grobman result as its deterministic counterpart. We prove that they can be linearized in a small random neighborhood of a hyperbolic fixed point of the linearized motion.

The structure of the paper is as follows. In section 1 the basic concepts of random cocycles, their generators, and cohomologies are briefly discussed. Section 2 is devoted to the main cohomology Theorem relating cocycles of stochastic and random differential equations by random stationary coordinate changes (Theorems 2.1, 2.2). In section 3, adapting a deterministic improvement of the Hartman Theorem by Palmer [16] to the stochastic setting, we give a short proof of a global (Theorem 3.1) and a local (Theorem 3.2) linearization result for random differential equations. Using cohomology, we carry Theorem 3.2 over to the setting of stochastic differential equations (Theorem 4.1).

1 Stochastic and random differential equations and cocycles

According to Arnold [1], random dynamical systems may be generated by a variety of different objects. Random matrices may as well act as generators as random or stochastic differential equations. To see that the second type of generator is just the continuous parameter version of the discrete time products of random matrices, let us briefly explain the notions. If $(\Omega, \mathcal{F}, P)$ is a stochastic basis with a $P-$preserving mapping $\theta : \Omega \to \Omega$, and $A : \Omega \to \mathbf{R}^{d\times d}$ is a random $d \times d-$matrix, we may define for

$n \in \mathbf{Z}_+, \omega \in \Omega$

$$\phi(n,\omega) = A(\theta^{n-1}\omega) \circ A(\theta^{n-2}\omega) \circ \cdots \circ A(\omega).$$

Then for $n, m \in \mathbf{Z}_+, \omega \in \Omega$

$$\begin{aligned}\phi(n+m,\omega) &= A(\theta^{n-1}\theta^m\omega) \circ \cdots A(\theta^m\omega) \circ A(\theta^{m-1}\omega) \circ \cdots A(\omega) \\ &= \phi(n,\theta^m\omega) \circ \phi(m,\omega),\end{aligned} \tag{1}$$

and trivially

$$\phi(0,\omega) = \mathrm{id}_{\mathbf{R}^d}. \tag{2}$$

So in case $\mathbf{T} = \mathbf{Z}_+$ and the semigroup of $P-$preserving mappings $(\theta_t)_{t\in\mathbf{Z}}$ is interpreted by $\theta_t = \theta^t$ for $t \in \mathbf{Z}$, our flow of products of random matrices ϕ is a random cocycle or random dynamical system in the sense of the following definition.

Definition 1.1 *Let* $\mathbf{T} \in \{\mathbf{Z}_+, \mathbf{Z}, \mathbf{R}_+, \mathbf{R}\}$ *and* $d \in \mathbf{N}$. *Let* $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{T}})$ *be a* metric dynamical system, *i. e. a stochastic basis endowed with a (semi)group of $P-$preserving maps. Then a map*

$$\Phi : \mathbf{T} \times \Omega \times \mathbf{R}^d \to \mathbf{R}^d$$

is called random dynamical system *or* random cocycle *on* $\mathbf{R}^d$ *if the following conditions are fulfilled:*

(i) Φ *is* $\mathcal{B}(\mathbf{T}) \otimes \mathcal{F} \otimes \mathcal{B}^d$*-measurable,*

(ii) for $s, t \in \mathbf{T}, \omega \in \Omega$ *we have*

$$\Phi(s+t,\omega) = \Phi(t,\theta_s\omega) \circ \Phi(s,\omega),$$

$$\Phi(0,\omega) = id_{\mathbf{R}^d}.$$

At places, we shall also refer to random cocycles as *random flows*, slightly abusing the standard terminology. Here and in the sequel we shall write $\Phi(t,\omega)$ or $\Phi_t(\omega)$ for the $(t,\omega)-$section of Φ.

Random dynamical systems induced by products of random matrices may be viewed slightly differently in the following random difference equation version. Setting

$$g : \Omega \times \mathbf{R}^d \to \mathbf{R}^d, (\omega, y) \mapsto (A - \mathrm{id}_{\mathbf{R}^d})(\omega, y),$$

we obtain the random dynamical system generated by the above products of translates of A by θ also as the solution flow generated by the random difference equation

$$y_{n+1} - y_n = g(\theta_n \cdot, y_n), \quad n \in \mathbf{Z}_+. \tag{3}$$

The continuous time version of this random dynamical system will then be generated by a metric dynamical system with a continuous time (semi)group of $P-$preserving transformations $\theta_t : \Omega \to \Omega$, $t \in \mathbf{R}\,(t \geq 0)$ and a random mapping $g : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ with some smoothness, e. g. local Lipschitz properties, in the spatial variable, through the *random differential equation*

$$dy_t = g(\theta_t \cdot, y_t)\, dt, \quad t \in \mathbf{R}\ (t \geq 0). \tag{4}$$

The random cocycle corresponding to such an equation is naturally induced by the flow of solutions of the differential equation (see Arnold, Scheutzow [3]).

The third canonical source of random dynamical systems is *stochastic differential equations.* For the smooth form of this object, one usually departs from the $m-$dimensional canonical Wiener space $(\Omega, \mathcal{F}, P)$. Here Ω is the space of continuous $\mathbf{R}^m-$valued functions on $\mathbf{R}$ endowed with the $\sigma-$algebra $\mathcal{F}$ of Borel sets for the topology of uniform convergence on compact subintervals of $\mathbf{R}$. Together with the *canonical group of time shifts* $\theta_t : \Omega \to \Omega, \omega \mapsto (s \mapsto \omega_{t+s} - \omega_t)$, $t \in \mathbf{R}$ $(t \geq 0)$, the canonical space creates a metric dynamical system. Given smooth vector fields $f_0, \cdots, f_m$ on $\mathbf{R}^d$, the stochastic differential equation in Stratonovich form

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i, \quad t \in \mathbf{R} \;\; (t \geq 0), \tag{5}$$

induces a flow of solutions which gives rise to a random cocycle indexed by $\mathbf{R}$ resp. $\mathbf{R}_+$.

Be the stochastic differential equations given in Stratonovich or Itô form, contrary to random differential equations, it is by no means trivial to prove the cocycle property of the induced flow, not even the flow property of the solutions itself. If it comes to asymptotic properties of cocycles generated by random differential equations resp. stochastic differential equations, this clear dichotomy of difficulties in the stochastic and analytical treatment becomes even more pronounced. The multiplicative ergodic theory of random differential equations often just requires an ω-by-ω extension of arguments available for deterministic differential equations. In contrast to this, the multiplicative ergodic theory of stochastic differential equations is essentially harder to handle, due to some incompatibility between the causality notions of stochastic analysis and Itô's calculus on the one hand, and the notions of ergodic theory on the other hand reflecting causality only in a rather restricted way: asymptotic notions involving both $\alpha-$ and $\omega-$limits for instance depend on information from the far past and future at the same time.

Here we propose a way out of this dilemma. It is based on the following notion investigated in [9], [10] for sde with special algebraic conditions to be fulfilled by the diffusion vector fields. Suppose from now on that the metric dynamical system on which our random dynamical systems are based, is composed of the $m-$dimensional canonical Wiener space with the canonical group of time shifts.

Definition 1.2 *Two random dynamical systems Φ and Ψ on $\mathbf{R}^d$ are called* conjugate, *if there exists a random homeomorphism $H : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ such that for all (ω, t) we have*

$$\Psi_t(\omega) = H(\theta_t\omega, \cdot) \circ \Phi_t(\omega) \circ H^{-1}(\omega, \cdot).$$

In this case H is called cohomology *of Φ and Ψ.*

A cohomology of two random dynamical systems may be considered a random coordinate change through which the dynamical behaviour described by one of them is translated into the dynamical behaviour described by the other. Intrinsic asymptotic notions of random dynamical systems such as Lyapunov exponents or random attractors should not be altered by coordinate changes providing cohomologies. Hence if e.g. the attractor of one of them is known, it should be easy to obtain an attractor for the other by simply mapping the former by means of the cohomology.

We shall prove in the following section that under very mild conditions on the vector fields of a stochastic differential equation there always exists a random vector field such that the random dynamical system generated by the rde related to this field is conjugate to the random dynamical system generated by the sde. This opens a new way to construct for example random attractors for sde given some knowledge on attractors of related rde. This idea has been used in Crauel, Flandoli [8], Keller, Schmalfuss [11] in very particular cases, and exploited more systematically in [9] and [10] under algebraic conditions concerning the Lie algebras generated by the diffusion vector fields.

2 The cohomology theorem

Let smooth vector fields $f_0, \cdots, f_m$ on $\mathbf{R}^d$ induce the stochastic differential equation

$$dx_t = f_0(x_t)\,dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i, \quad t \geq 0, \tag{6}$$

which generates the smooth random dynamical system Φ on $\mathbf{R}^d$. Then the following decomposition of its flow is well known from Bismut, Michel [5], [6], and has been used in a number of papers. Given a differentiable map $g : \mathbf{R}^d \to \mathbf{R}^d$, we denote in the sequel by $\frac{\partial}{\partial x} g$ the Jacobian of g. Let H be the random dynamical system generated by the stochastic differential equation induced by the diffusion vector fields alone

$$dX_t = \sum_{i=1}^m f_i(X_t) \circ dW_t^i, \quad t \geq 0. \tag{7}$$

Denote by Ψ the flow generated by the non-autonomous random differential equation

$$dY_t = (\frac{\partial}{\partial x} H_t)^{-1}(\cdot, Y_t)\, f_0(H_t(\cdot, Y_t))\,dt, \quad t \geq 0. \tag{8}$$

Then, according to the Itô-Ventzell formula, we have

$$\begin{aligned} d(H_t(\cdot, \Psi_t(\cdot, x))) &= \frac{\partial}{\partial x} H_t(\cdot, \Psi_t(\cdot, x)) \circ d\Psi_t(\cdot, x) + dH_t(\cdot, y)|_{y=\Psi_t(\cdot, x)} \\ &= f_0(H_t(\cdot, \Psi_t(\cdot, x)))\,dt + \sum_{i=1}^m f_i(H_t(\cdot, \Psi_t(\cdot, x))) \circ dW_t^i. \end{aligned}$$

Therefore, by uniqueness of solutions

$$\Phi_t = H_t \circ \Psi_t = H_t \circ \Psi_t \circ \mathrm{id}_{\mathbf{R}^d}^{-1}. \tag{9}$$

Stated alternatively, Φ and Ψ are related by the coordinate changes $\mathrm{id}_{\mathbf{R}^d}$ and H_t at time $t \geq 0$. This, of course, usually fails to be a cohomology, since $H_t(\omega)$ will in general be very different from $H_0(\theta_t\omega)$, for $\omega \in \Omega$, $t > 0$. But the flow decomposition idea points into the right direction. The random cocycle generated by (7) just lacks the property of being *stationary*, expressed by the equation

$$H_t(\omega) = H_0(\theta_t\omega), \quad \omega \in \Omega, t \geq 0.$$

To obtain this property, we shall in the sequel modify the sde induced by the diffusion vector fields so that it generates a stationary flow of diffeomorphisms of $\mathbf{R}^d$. To catch the idea of how this could be done, let us briefly re-examine the well known Doss-Sussmann flow decomposition of a simple stochastic differential equation. Suppose in addition to the above smoothness hypotheses that the diffusion vector fields commute, i.e. in the usual sense of differential geometry that

$$[f_i, f_j] = 0, \quad 1 \leq i, j \leq m, i \neq j.$$

Then the partial differential equation

$$\begin{aligned} \frac{\partial \phi}{\partial w_i}(w_1, \cdots, w_m, x) &= f_i(\phi(w_1, \cdots, w_m, x)), \quad 1 \leq i \leq m, \\ \phi(0, x) = x, & \end{aligned}$$

possesses a smooth solution $\phi : \mathbf{R}^m \times \mathbf{R}^d \to \mathbf{R}^d$. To obtain the flow of diffeomorphisms solving (6), it is now enough to take the random flow $H_t = \phi(W_t, \cdot), t \in \mathbf{R}$. And in this setting it is easy to make this non-stationary flow stationary. We just replace the Wiener process by its stationary companion, the Ornstein-Uhlenbeck process given by

$$Z_t^i = e^{-t} \int_{-\infty}^t e^s \, dW_s^i, \quad t \in \mathbf{R}, \ 1 \leq i \leq m. \tag{10}$$

Then redefining H by

$$H_t = \phi(Z_t, \cdot), t \in \mathbf{R},$$

will just provide the right object, a cohomology. Which objects does this cohomology relate? On the one hand, we still have the flow Φ of the original equation (6). But on the other hand, the corresponding generator is altered by a correcting drift, due to the well known relation

$$dZ_t = dW_t - Z_t \, dt.$$

In fact, applying Itô's formula produces

$$dX_t = \sum_{i=1}^m f_i(X_t) \circ dW_t^i - \sum_{i=1}^m Z_t^i f_i(X_t) \, dt.$$

This drift term has to be taken into account in an analogue of (8) given by

$$dY_t = (\frac{\partial}{\partial x} H_t)^{-1}(\cdot, Y_t) \, [f_0(H_t(\cdot, Y_t)) + \sum_{i=1}^m Z_t^i \, f_i(H_t(\cdot, Y_t))] \, dt, \quad t \geq 0, \tag{11}$$

so that we will obtain the sde generating Φ as its random cocycle. Defining

$$g(\cdot, y) = (\frac{\partial}{\partial x} H_0)^{-1}(\cdot, y) \, [f_0(H_0(\cdot, y)) + \sum_{i=1}^m Z_0^i \, f_i(H_0(\cdot, y))], \quad y \in \mathbf{R}^d,$$

thus solves the problem of finding a random differential equation and a cohomology such that the flow of our original equation (6) and the flow of the random differential equation $dy_t = g(\theta_t \cdot, y_t)\, dt$ are conjugate, in the particular case of algebraic simplicity of the diffusion vector fields. At the same time, it suggests a two step algorithm which, appropriately modified, could give the solution in general:

(i) find a stationary solution of an sde containing the diffusion part (7), and determine the cohomology as its flow H,

(ii) given H, find the random vector field g which determines the conjugate flow Ψ through the random differential equation $dy_t = g(\theta_t \cdot, y_t)\, dt$.

Suppose now that we are back to the general setting, i.e. the commutation property of the diffusion vector fields is possibly not satisfied. Now of course there is in general no integral form as above into which we just have to plug a random process R to get a flow generated by the sde containing only the diffusion vector fields with R as driving noise. In order to carry out step 1 of our algorithm, we therefore have to think of a different way to obtain a stationary flow of solutions of an sde containing (7). To get an idea, let us briefly think about the relationship between the scalar Wiener process and the stationary scalar Ornstein-Uhlenbeck process. In the setting of our sde, this corresponds to $m = d = 1, f_0 = 0, f_1 = 1$. To obtain the flow of the diffusion part which now is identical to the whole equation, we have to solve the trivial integral equations

$$X_t^x = x + W_t, \quad t \in \mathbf{R},\ x \in \mathbf{R}. \tag{12}$$

To obtain a stationary flow of an sde containing the right diffusion part, thinking of the moving average description of the stationary OU process, we may solve the non canonical integral equations

$$h_t^x = x + e^{-t} \int_{-\infty}^{t} e^s \, dW_s, \quad t \in \mathbf{R},\ x \in \mathbf{R}. \tag{13}$$

To carry this idea further, in the general setting, instead of solving the sde (7), we may therefore try to solve the non canonical sde

$$h_t^x = x + e^{-t} \sum_{i=1}^{m} \int_{-\infty}^{t} e^s \, f_i(h_s^x) \circ dW_t^i, \quad t \in \mathbf{R},\ x \in \mathbf{R}^d. \tag{14}$$

To make sense of the sde (14), we shall introduce a free parameter $\tau \in \mathbf{R}$ for the averaging factor, and investigate the parametrized sdes

$$h_t^{x,\tau} = x + e^{-\tau} \sum_{i=1}^{m} \int_{-\infty}^{t} e^s \, f_i(h_s^{x,\tau}) \circ dW_t^i, \quad t, \tau \in \mathbf{R},\ x \in \mathbf{R}^d, \tag{15}$$

by means of the usual arguments of stochastic calculus. Its solutions will provide a random cocycle parametrized by the pair (t, τ), and finally we will have to set $t = \tau$ in order to obtain the stationary cocycle H we are looking for. The only aspect that might seem bothering is the infinite interval of integration appearing in (15). As we shall see, a simple change of time scale will cast us back to the usual setting. To obtain the desired properties of the flow of parametrized systems as in (15), we will need some notation borrowed from Kunita [13]. For $m \in \mathbf{Z}_+, \delta > 0$ denote by $\mathcal{C}_b^{m,\delta}$ the set of functions $f : \mathbf{R}^d \to \mathbf{R}^d$ which possess partial derivatives $D^\alpha f$ of order up to $|\alpha| \leq m$, are linearly bounded and for which the derivatives of order m are $\delta-$Hölder continuous, i. e. for which the following inequality holds

$$\sup_{r\in\mathbf{R}^d} \frac{|f(x)|}{1+|x|} + \sum_{|\alpha|=1}^{m} \sup_{x\in\mathbf{R}^d} |D^\alpha f(x)| + \sum_{|\alpha|=m} \sup_{x,y\in\mathbf{R}^d, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta} < \infty.$$

In this setting, the following slight generalization of well known results about the existence of global flows holds.

Proposition 2.1 *Let $\delta > 0$, and suppose $f_0, \cdots, f_m \in \mathcal{C}_b^{1,\delta}$. Then for any $\alpha \in \mathbf{R}$ the Itô stochastic differential equation*

$$dx_t = \alpha[f_0(x_t)\,dt + \sum_{i=1}^{m} f_i(x_t)\,dW_t^i] \tag{16}$$

generates a global flow of diffeomorphisms Φ^α which is differentiable in the parameter α.

Proof:

We shall in fact prove the existence of a global flow of diffeomorphisms on the space $\mathbf{R}^d \times \mathbf{R}$ which is at least differentiable in the second coordinate. Formally, this is done by enlarging our sde by the equation $d\alpha_t = 0$. In this enlarged equation the vector fields $(x,\alpha) \mapsto \alpha\, f_i(x), 0 \leq i \leq m$, are only locally Lipschitz. We therefore know that there exists a local flow of diffeomorphisms (see Kunita [13]) Φ on $\mathbf{R}^d \times \mathbf{R}$ which is differentiable in α. It remains to verify that this flow is indeed global. This will be done by using the continuity criterion for stochastic processes due to Kolmogorov, and the particular structure of the vector fields.

Denote by $X^{x,\alpha}$ the solution of the enlarged (16) with initial conditions (x,α). Fix an arbitrary compact set $K \subset \mathbf{R}^d \times \mathbf{R}$, $T > 0$, and $p \geq 2$. Note first that there exists a constant c_1 such that for $(x,\alpha), (y,\alpha) \in K$, $0 \leq i \leq m$ we have

$$\begin{aligned} |\alpha f_i(x) - \alpha f_i(y)| &\leq c_1\,|x-y|, \\ |\alpha\, f_i(x)| &\leq c_1(1+|x|). \end{aligned}$$

Hence, the usual combination of Burkholder's, Jensen's and Hölder's inequalities provides the following estimate for $(x,\alpha) \in K$ with a constant c_2 independent of K

$$E(\sup_{0\leq t\leq T} |X_t^{x,\alpha}|^p) \leq c_2\,[|x|^p + \int_0^T E(\sup_{0\leq t\leq s} |X_t^{x,\alpha}|^p)\,ds].$$

Hence Gronwall's lemma is applicable and we obtain that

$$M = \sup_{(x,\alpha)\in K} E(\sup_{0\leq t\leq T} |X_t^{x,\alpha}|^p) < \infty.$$

Note next that for $(x,\alpha),(y,\beta)\in K, t\in[0,T]$ we may write (abbreviating $dW_t^0=dt$)

$$\begin{aligned}X_t^{x,\alpha}-X_t^{y,\beta} &= x-y+\alpha\sum_{i=0}^m\int_0^t[f_i(X_s^{x,\alpha})-f_i(X_s^{y,\beta})]dW_s^i\\ &+ (\beta-\alpha)\sum_{i=0}^m\int_0^t f_i(X_s^{y,\beta})\,dW_s^i.\end{aligned}$$

A slightly different estimation from the one above then yields a constant c_3 independent of K but depending on M such that

$$E(\sup_{0\le t\le T}|X_t^{x,\alpha}-X_t^{y,\beta}|^p)\le c_3\,[|x-y|^p+|\beta-\alpha|^p+\int_0^T E(\sup_{0\le t\le s}|X_s^{x,\alpha}-X_s^{y,\beta}|^p\,ds)].$$

Another application of Gronwall's lemma produces a constant c_4 such that for $(x,\alpha),(y,\beta)\in K, T>0$ we have

$$E(\sup_{0\le t\le T}|X_t^{x,\alpha}-X_t^{y,\beta}|^p)\le c_4\,[|x-y|^p+|\beta-\alpha|^p].$$

K and T being arbitrary, Kolmogorov's continuity criterion therefore implies that our flow Φ on $\mathbf{R}^d\times\mathbf{R}$ is global. □

We are ready to formulate and prove the main result of this section. It describes the cohomology through a stochastic differential equation. Denote in what follows the components of vector fields by upper indices.

Theorem 2.1 *Let $\delta>0$. Suppose that $f_1,\cdots,f_m\in\mathcal{C}_b^{2,\delta}$, and $\sum_{i=1}^m\sum_{j=1}^d f_i^j\,\frac{\partial f_i}{\partial x_j}\in C_b^{2,\delta}$. Then there exists a random flow of diffeomorphisms Φ on $\mathbf{R}^d\times\mathbf{R}$ such that for any $(x,\tau)\in\mathbf{R}^d\times\mathbf{R}$ the process $\Phi(x,\tau)$ satisfies the stochastic integral equation*

$$\Phi_t(x,\tau)=x+e^{-\tau}\sum_{i=1}^m\int_{-\infty}^t e^s\,f_i(\Phi_s(x,\tau))\circ dW_s^i,\quad t\in\mathbf{R}. \tag{17}$$

Let

$$H_t=\Phi_t(\cdot,t), \tag{18}$$

$$\Gamma_t=\frac{\partial}{\partial\tau}\Phi_t(\cdot,t),\quad t\in\mathbf{R}. \tag{19}$$

Then H is a stationary cocycle of diffeomorphisms on $\mathbf{R}^d$, Γ a stationary random vector field on $\mathbf{R}^d$, and for any $x\in\mathbf{R}^d$ the processes $H(x)$ and $\Gamma(x)$ satisfy the sde

$$dH_t(x)=\sum_{i=1}^m f_i(H_t(x))\circ dW_t^i+\Gamma_t(x)\,dt,\quad t\in\mathbf{R}. \tag{20}$$

Moreover, for $x\in\mathbf{R}^d$, $\Gamma(x)$ satisfies the stochastic integral equation

$$\begin{aligned}\Gamma_t(x) &= -(H_t(x)-x)+e^{-t}\sum_{i=1}^m\int_{-\infty}^t e^s\frac{\partial}{\partial x}f_i(H_s(x))\Gamma_s(x)\,\circ dW_s^i \qquad (21)\\ &= -e^{-t}\sum_{i=1}^m\int_{-\infty}^t e^s f_i(H_s(x))\,\circ dW_s^i\\ &\quad +e^{-t}\sum_{i=1}^m\int_{-\infty}^t e^s\frac{\partial}{\partial x}f_i(H_s(x))\Gamma_s(x)\,\circ dW_s^i,\quad t\in\mathbf{R}.\end{aligned}$$

Proof:

For $(x,\tau) \in \mathbf{R}^d$ let us consider the stochastic integral equation

$$h_t^{x,\tau} = x + e^{-\tau} \sum_{i=1}^m \int_{-\infty}^t e^s f_i(h_s^{x,\tau}) \circ dW_s^i, \quad t \in \mathbf{R}. \tag{22}$$

To be able to use the result of Proposition 2.1, we rescale time for the processes involved in the following way. For $t > 0, x \in \mathbf{R}^d, \tau \in \mathbf{R}, 1 \leq i \leq m$, let

$$\begin{aligned} B_t^i &= \int_{-\infty}^{\frac{1}{2}\ln 2t} e^s \, dW_s^i, \\ g_t^{x,\tau} &= h_{\frac{1}{2}\ln 2t}^{x,\tau}. \end{aligned}$$

Then $B = (B^1, \cdots, B^m)$ is an $m-$dimensional Wiener process indexed by $\mathbf{R}_+$, and (22) is equivalent to the transformed stochastic integral equation

$$g_t^{x,\tau} = x + e^{-\tau} \sum_{i=1}^m \int_0^t f_i(g_s^{x,\tau}) \circ dB_s^i, \quad t > 0. \tag{23}$$

Setting $\alpha = e^{-\tau}$, we are therefore in the situation of Proposition 2.1. We conclude that there exists a random flow of diffeomorphisms Ψ of $\mathbf{R}^d \times \mathbf{R}$ such that for any $(x,\tau) \in \mathbf{R}^d \times \mathbf{R}$ the process $\Psi(x,\tau)$ satisfies the stochastic integral equation

$$\Psi_t(x,\tau) = x + e^{-\tau} \sum_{i=1}^m \int_0^t f_i(\Psi_s(x,\tau)) \circ dB_s^i, \quad t > 0. \tag{24}$$

It is therefore clear that

$$\Phi_t = \Psi_{\frac{1}{2}e^{2t}}, \quad t \in \mathbf{R},$$

defines the random flow of diffeomorphisms satisfying (17). Now let

$$H_t = \Phi_t(\cdot, t), \quad t \in \mathbf{R}.$$

Due to the smoothness properties of Φ, choosing $x \in \mathbf{R}^d$, we may apply a version of the Itô-Ventzell formula to get

$$\begin{aligned} dH_t(x) &= d\Phi_t(x,\tau)|_{\tau=t} + \frac{\partial}{\partial \tau}\Phi_t(x,t)dt \\ &= \sum_{i=1}^m f_i(H_t(x)) \circ dW_t^i + \frac{\partial}{\partial \tau}\Phi_t(x,t)dt, \quad t \in \mathbf{R}. \end{aligned} \tag{25}$$

It is therefore clear how Γ has to be chosen. The sde it satisfies is straightforward, and differentiation is justified by our hypotheses.

It remains to prove stationarity. For this purpose let $s, t \in \mathbf{R}, x \in \mathbf{R}^d$. Then

$$\begin{aligned} H_s(\theta_t \cdot, x) &= x + \sum_{i=1}^m [e^{-s} \int_{-\infty}^s e^u f_i(H_u(\cdot, x)) \circ dW_u^i] \circ \theta_t \\ &= x + \sum_{i=1}^m e^{-s} \int_{-\infty}^s e^u f_i(H_u(\theta_t \cdot, x)) \circ dW_{u+t}^i \\ &= x + \sum_{i=1}^m e^{-(s+t)} \int_{-\infty}^{s+t} e^v f_i(H_{v-t}(\theta_t \cdot, x)) \circ dW_v^i. \end{aligned}$$

By uniqueness of solutions, and by smoothness in x, we therefore obtain the equation

$$H_s(\theta_t\cdot,\cdot) = H_{s+t}(\cdot,\cdot). \tag{26}$$

Via a perfection argument (see Lederer [14], Satz 2.8, Arnold, Scheutzow [3]), stationarity is obtained from (26). The argument needed to show that Γ is stationary is quite similar. □

This completes at the same time the first step in our two-step algorithm for solving the cohomology problem for (6). The second step is now very easy.

Theorem 2.2 *Let $\delta > 0$. Suppose that $f_0 \in \mathcal{C}_b^{1,\delta}$, $f_1, \cdots, f_m \in \mathcal{C}_b^{2,\delta}$, and $\sum_{i=1}^m \sum_{j=1}^d f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{2,\delta}$. Let H and Γ be given by Theorem 2.1. Define the random vector field $g : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ by*

$$g(\cdot, y) = \frac{\partial}{\partial x} H_0^{-1}(y)\,[f_0(H_0(y)) + \Gamma_0(y)]. \tag{27}$$

Then

(i) the random differential equation

$$dy_t = g(\theta_t\cdot, y_t)\,dt, \quad t \in \mathbf{R}, \tag{28}$$

generates a global random cocycle of diffeomorphisms Ψ,

(ii) the sde

$$dx_t = f_0(x_t)\,dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i, \quad t \in \mathbf{R}, \tag{29}$$

generates a global cocycle of diffeomorphisms Φ.

Φ and Ψ are conjugate with cohomology H_0, i.e. we have for $t \in \mathbf{R}, \omega \in \Omega$

$$\Phi_t(\omega) = H_0(\theta_t\omega,\cdot) \circ \Psi_t(\omega) \circ H_0(\omega,\cdot)^{-1}. \tag{30}$$

Proof:
According to Arnold [1], (29) generates a global cocycle of diffeomorphisms of $\mathbf{R}^d$. The smoothness properties of g allow to conclude only that (28) generates a local cocycle, say Ψ. But by the Itô-Ventzell formula, we can show that

$$\Lambda_t = H_0(\theta_t\cdot,\cdot) \circ \Psi_t \circ H_0^{-1}, \quad t \in \mathbf{R},$$

is another cocycle of diffeomorphisms generated by (29). But this cocycle must, by uniqueness of solutions, coincide with Φ, which is global. Hence Ψ is global as well, and the cohomology is established. □

To illustrate our cohomology Theorem, let us briefly discuss some examples. Let us first return to the simple case of commuting diffusion vector fields, in which, as we mentioned in the outset, the random cohomology can be explicitly calculated in a generalization of the Doss-Sussmann representation.

Example 1:
Let $A_0, \cdots, A_m \in \mathbf{R}^{d\times d}$ be such that $[A_i, A_j] = 0$ for $1 \le i, j \le m$ and

$$dx_t = A_0\, x_t\, dt + \sum_{i=1}^m A_i\, x_t \circ dW_t^i.$$

Let Z be the stationary solution of the $m-$dimensional Langevin equation

$$dZ_t = dW_t - Z_t\, dt.$$

Then, as explained at the beginning of the section, the cohomology H will be given as the cocycle generated by the solutions of the sde induced by the diffusion part alone with W replaced by Z

$$dX_t = \sum_{i=1}^m A_i\, X_t \circ dZ_t^i.$$

This cocycle is easy to determine. Due to the commutation property of the matrices $A_1, \cdots, A_m$ the system of linear differential equations

$$\frac{\partial \phi}{\partial z_i}(z, x) = A_i\, \phi(z, x), \quad z \in \mathbf{R}^m, x \in \mathbf{R}^d, \quad 1 \le i \le m,$$

$$\phi(0, x) = x, \quad x \in \mathbf{R}^d,$$

possesses the fundamental solution

$$\phi(z, \cdot) = \exp(\sum_{i=1}^m A_i\, z_i).$$

Hence

$$H_t = \phi(Z_t, \cdot) = \exp(\sum_{i=1}^m A_i\, Z_t^i) = \exp(\sum_{i=1}^m A_i\, Z_0^i) \circ \theta_t = H_0 \circ \theta_t.$$

According to (11) the random vector field of the cohomologous rde is given by

$$g(\omega, y) = H_0^{-1}(\omega)\, A_0\, H_0(\omega)\, y + \sum_{i=1}^m A_i\, y\, Z_0^i(\omega),$$

$\omega \in \Omega, y \in \mathbf{R}^d$.

Let us next consider the case of affine vector fields, with commuting linear parts. Here, the cohomology is almost as simple as in the preceding case.

Example 2:
Let $A_0, \cdots, A_m \in \mathbf{R}^{d\times d}, b_1, \cdots, b_m \in \mathbf{R}^d$ be such that $[A_i, A_j] = 0$ for $1 \le i, j \le m$ and consider

$$dx_t = A_0\, x_t\, dt + \sum_{i=1}^m (A_i\, x_t + b_i) \circ dW_t^i.$$

As before, let Z be the stationary solution of the Langevin equation

$$dZ_t = dW_t - Z_t\,dt,$$

and

$$G_0 = \exp(A_1\,Z_0^1 + \cdots + A_m\,Z_0^m).$$

Denoting by A^{-1} the pseudo-inverse of a matrix A, we let

$$H_0(\cdot, x) = G_0(\cdot, x) + \sum_{i=1}^m A_i^{-1}(\exp(A_i\,Z_0^i) - I)\,b_i, \quad x \in \mathbf{R}^d.$$

This defines our random cohomology by

$$H_t = H_0 \circ \theta_t, \quad t \in \mathbf{R}.$$

According to (11), the random vector field of the cohomologous rde is given by

$$g(\omega, y) = H_0^{-1}(\omega)\,A_0\,H_0(\omega)\,y + \sum_{i=1}^m H_0^{-1}(\omega)\,A_i\,H_0(\omega)\,Z_0^i(\omega)\,y,$$

$\omega \in \Omega, y \in \mathbf{R}^d$.

In a third example, we allow the Lie algebra generated by the linear diffusion vector fields to be slightly more complicated. We shall see that the computation of the associated cohomology still gives an explicit result, though it is more involved this time. We follow an algorithm developed in [10].

Example 3:
The following is a simplification of the well known noisy Duffing-van der Pol oscillator with independent noise sources coupled to the position and velocity components. Its cohomology to an rde has been used in [10] to show the existence of a random attractor for this system. In the usual position-velocity coordinates the simplified system has the following description

$$dx_t = f_0(x_t)\,dt + A_1\,x_t \circ dW_t^1 + A_2\,x_t \circ dW_t^2,$$

with the matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and a nonlinear vector field f_0 for which the system is *conservative*, i.e. possesses global solutions for any initial vector. The algorithm proposed in [10] rests upon the complexity of the Lie algebra $\mathcal{L}$ generated by A_1 and A_2. We define recursively, denoting by

$$[A, B] = AB - BA$$

the Lie bracket of the matrices A and B in $\mathbf{R}^{d\times d}$,

$$\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}], \quad \text{and} \quad \mathcal{L}_{n+1} = [\mathcal{L}_n, \mathcal{L}_n], \quad n \in \mathbf{N}.$$

The Lie algebra is called *solvable*, if for some $n \in \mathbf{N}$ we have $\mathcal{L}_n = 0$. In the case of the two matrices appearing in our sde we have

$$\mathcal{L} = \mathrm{span}\{A_1, A_2\}, \quad \text{and} \quad [A_2, A_1] = A_1.$$

Hence

$$\mathcal{L}_1 = \mathrm{span}\{A_1\}, \quad \mathcal{L}_2 = 0.$$

So $\mathcal{L}$ is solvable, but not nilpotent. Now according to the degree of solvability 2 of the underlying Lie algebra we construct the cohomology in two steps essentially similar to the one presented in Example 1. First write the sde in the form

$$dx_t \;=\; B_t^0(x_t)dt + \sum_{i=1}^{2} A_i^0 x_t \circ dZ_i^0(t), \tag{31}$$

and denote its induced flow by ϕ^0, where the stationary vector field B^0 is defined by

$$B_t^0(x) \;=\; f_0(x) + \sum_{j=1}^{2} A_j^0 \; Z_j^0(t)\, x, \tag{32}$$

$A_i^0 = A_i, i = 1, 2$, and $Z^0 = Z$ is the 2-dimensional stationary Ornstein-Uhlenbeck process. Define the linear stationary vector field

$$C_t^0 \;=\; \sum_{i=1}^{2} A_i^0 \, Z_i^0(t), \quad t \in \mathbf{R}, \tag{33}$$

let

$$H_t^0 \;=\; \exp(-C_t^0),$$

and define the flow ϕ^1 by

$$\phi_t^1 \;=\; H_t^0 \, \phi_t^0 \, (H_0^0)^{-1}. \tag{34}$$

Now if A_1, A_2 were commuting, according with the results of Example 1, we would be able to conclude that ϕ^1 has a vanishing diffusion part, so that H^0 would give an appropriate cohomology. In this case we would be able to stop the algorithm after this step. Since they do not commute, we write for convenience the flow of the first step in the form

$$d\phi_t^0(x) = B_t^0(\phi_t^0(x))\, dt + \circ dC_t^0 \, \phi_t^0(x),$$

and derive the sde determining ϕ^1 in the form

$$\begin{aligned} d\phi_t^1(x) \;&=\; \circ d\Phi_t^0 \; \phi_t^0 \; (\Phi_0^0)^{-1}(x) + \Phi_t^0 \; \circ d\phi_t^0 \; (\Phi_0^0)^{-1}(x) \\ &=\; \left[-\Phi_t^0 \circ dC_t^0 - \int_0^1 s \; e^{-sC_t^0} \; [C_t^0, \circ dC_t^0] e^{sC_t^0} \; ds \; \Phi_t^0\right] \; \phi_t^0 \; (\Phi_0^0)^{-1}(x) \\ &\quad + \Phi_t^0 \; \left[B_t^0 \phi_t^0 \; dt + \circ dC_t^0 \; \phi_t^0\right] \; (\Phi_0^0)^{-1}(x) \\ &=\; -\int_0^1 s \; e^{-sC_t^0} \; [C_t^0, \circ dC_t^0] e^{sC_t^0} \; ds \; \phi_t^1 + \Phi_t^0 B_t^0 \phi_t^0 (\Phi_0^0)^{-1} \; dt \\ &=\; -\int_0^1 s \; e^{-sC_t^0} \; [C_t^0, \circ dC_t^0] e^{sC_t^0} \; ds \; \phi_t^1 + \Phi_t^0 B_t^0 (\Phi_t^0)^{-1} \; \phi_t^1 \; dt. \end{aligned} \tag{35}$$

Here the second equation is based on the simple derivation formula for smoothly parametrized exponentials $e^{A(\lambda)}$ (see [10], Lemma 3.1.)

$$\frac{d}{d\lambda}e^{A(\lambda)} = e^{A(\lambda)}A'(\lambda) - \int_0^1 s\, e^{sA(\lambda)}[A(\lambda), A'(\lambda)]\, e^{-sA(\lambda)}\, ds\, e^{A(\lambda)}.$$

Next consider the process Γ^1 defined by the Stratonovich exponential

$$\circ d\Gamma_t^1 = -\int_0^1 s\, e^{-sC_t^0}\, [C_t^0, \circ dC_t^0] e^{sC_t^0}\, ds,$$

which possesses stationary increments but is not stationary. We make it stationary by defining

$$C_t^1 = e^{-t}\int_{-\infty}^t e^s \circ d\Gamma_s^1, \quad t \in \mathbf{R}. \tag{36}$$

The obvious equation

$$\circ d\Gamma_t^1 = \circ dC_t^1 + C_t^1\, dt$$

then allows to write (35) in the suggestive form

$$d\phi_t^1(x) = B_t^1\, \phi_t^1(x)\, dt + \circ dC_t^1\, \phi_t^1(x), \tag{37}$$

with the stationary vector field

$$B_t^1 = \Phi_t^0\, B_t^0\, (\Phi_t^0)^{-1} + C_t^1,$$

$t \in \mathbf{R}$. Now the recursive step is evident. We have to define

$$H_t^1 = \exp(-C_t^1), \quad t \in \mathbf{R},$$

which is stationary by definition, and set

$$\phi_t^2 = H_t^1\, \phi_t^1\, (H_0^1)^{-1}, \quad t \in \mathbf{R}. \tag{38}$$

Then due to $\mathcal{L}_2 = 0$, the diffusion part of ϕ^2 has to vanish, our algorithm can be stopped here, and we see that

$$H_t = H_t^1 \circ H_t^0, \quad t \in \mathbf{R},$$

is a suitable cohomology of our sde to an rde generated by the random vector field

$$g(\cdot, y) = H_0^1 \circ H_0^0\, f_0((H_0^1 \circ H_0^0)^{-1} y) + H_0^0\, C_0^0\, (H_0^0)^{-1}\, y + H_0^1 \circ H_0^0\, C_0^1\, (H_0^1 \circ H_0^0)^{-1}\, y,$$

$y \in \mathbf{R}^2$.

Let us finally calculate H. Since $C_0^0 = Z_1^0(0)A_1 + Z_2^0(0)A_2$, we have

$$H_0^0 = e^{-C_0^0} = \begin{bmatrix} 1 & 0 \\ v_2 & v_1 \end{bmatrix},$$

where

$$v_1 = e^{-Z_2^0(0)}, \quad v_2 = -\frac{Z_1^0(0)}{Z_2^0(0)}(1 - e^{-Z_2^0(0)}).$$

To compute C^1, let us start with the following special case of (36)

$$\begin{aligned} C_t^1 &= -e^{-t} \int_{-\infty}^t e^u \int_0^1 s\, e^{-sC_u^0} [C_u^0, \circ dC_u^0] e^{sC_u^0}\, ds \\ &= -e^{-t} \int_{-\infty}^t e^u \int_0^1 s\, e^{-s \sum_{l=1}^2 Z_l^0(u)\, A_l} [A_1, A_2] e^{s \sum_{l=1}^2 Z_l^0(u)\, A_l}\, ds \circ da_{12}(u), \end{aligned} \tag{39}$$

where

$$\circ da_{12}(u) = Z_1^0(u) \circ dZ_2^0(u) - Z_2^0(u) \circ dZ_1^0(u) = Z_1^0(u) dZ_2^0(u) - Z_2^0(u) dZ_1^0(u)$$

is the differential of the *Ornstein-Uhlenbeck area* process corresponding to Z_1^0 and Z_2^0. In our special case, this gives

$$\begin{aligned} &\int_0^1 s\, e^{-s \sum_{l=1}^2 Z_l^0(u)\, A_l} [A_1, A_2] e^{s \sum_{l=1}^2 Z_l^0(u)\, A_l}\, ds \\ &= \int_0^1 s \begin{bmatrix} 1 & 0 \\ v_2(s) & v_1(s) \end{bmatrix} A_1 \begin{bmatrix} 1 & 0 \\ -\frac{v_2(s)}{v_1(s)} & \frac{1}{v_1(s)} \end{bmatrix} ds \\ &= \frac{1}{Z_2^0(u)^2} [1 - e^{-Z_2^0(u)} - Z_2^0(u)\, e^{-Z_2^0(u)}]\, A_1. \end{aligned}$$

Here $v_1(s) = e^{-s\, Z_2^0(u)}$, $v_2(s) = \frac{Z_1^0(u)}{Z_2^0(u)}(1 - e^{-s\, Z_2^0(0)})$. Consequently, noting

$$v_3 = -\int_{-\infty}^0 e^u \frac{1}{Z_2^0(u)^2} [1 - e^{-Z_2^0(u)} - Z_2^0(u)\, e^{-Z_2^0(u)}] \circ da_{12}(u),$$

we arrive at the equations

$$C_0^1 = \begin{bmatrix} 0 & 0 \\ -v_3 & 0 \end{bmatrix}, \quad H_0^1 = e^{-C_0^1} = \begin{bmatrix} 1 & 0 \\ v_3 & 1 \end{bmatrix}.$$

Therefore, finally, the cohomology is given by

$$H_0 = H_0^1 \circ H_0^0 = \begin{bmatrix} 1 & 0 \\ v_2 + v_3 & v_1 \end{bmatrix}, \quad H_t = H_0 \circ \theta_t, \quad t \in \mathbf{R}.$$

3 Local linearization for random differential equations

A famous theorem due to Hartman and Grobman states that deterministic dynamical systems can be linearized in the vicinity of hyperbolic points. More formally, suppose $f : \mathbf{R}^d \to \mathbf{R}^d$ is a smooth function, inducing the autonomous differential equation

$$dx_t = f(x_t)\, dt, \tag{40}$$

with a global flow Φ of diffeomorphisms of $\mathbf{R}^d$. Suppose further that the fixed point 0 of f is *hyperbolic*, i.e. that

$$A = \frac{\partial f}{\partial x}(0)$$

possesses only eigenvalues with nonvanishing real parts, and let Ψ be the linear flow associated with the linear differential equation

$$dy_t = A\, y_t\, dt. \tag{41}$$

The theorem states that there exists a (usually non-linear) homeomorphic coordinate change $h : \mathbf{R}^d \to \mathbf{R}^d$ such that *locally* Φ and Ψ are conjugate via h:

$$\Phi_t = h \circ \Psi_t \circ h^{-1}, \quad \text{for } |t| \text{ sufficiently small.} \tag{42}$$

Generalizations of this theorem to the setting of random or stochastic differential equations are clearly conceivable. In the first case one might consider, on a basic metric dynamical system, a smooth random mapping $f : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ possessing 0 as a fixed point, with a random Jacobian A at 0 which is hyperbolic, i.e. all of its Lyapunov exponents do not vanish. The equations become

$$dx_t = f(\theta_t \cdot, x_t)\, dt, \tag{43}$$

$$dy_t = A(\theta_t \cdot)\, y_t\, dt. \tag{44}$$

In the second case, on the canonical metric dynamical system induced by Wiener space discussed above, we may consider smooth vector fields $f_0, \cdots, f_m : \mathbf{R}^d \to \mathbf{R}^d$, all having 0 as a fixed point, and with Jacobians at 0 given by $A_0, \cdots, A_m$ respectively. This leads to the equations

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i, \tag{45}$$

$$dy_t = A_0\, y_t\, dt + \sum_{i=1}^m A_i\, y_t \circ dW_t^i. \tag{46}$$

The hyperbolicity condition in this case has to be formulated in terms of the Lyapunov spectrum of (46).

In this and the following section, we shall prove that random dynamical systems generated by stochastic differential equations can be locally linearized in the vicinity of hyperbolic points. This way we generalize the theorem due to Hartman and Grobman, and close a gap which was open since about 10 years. To complete this task, we will exemplify how the cohomology theorem typically can be brought to work. In a first step, we will show in the present section how arguments valid in the deterministic setting can be generalized to the neighboring setting of random differential equations. As indicated above, the multiplicative ergodic theorem (MET) of Oseledets will play a key role (see Arnold [1], Oseledets [15]). This theorem provides the random counterparts of eigenvalues and the corresponding eigenspaces associated with autonomous linear equations such as (41): Lyapunov exponents and Oseledets spaces play analogous roles for the investigation of asymptotic properties of trajectories of linear random or stochastic differential equations such as (44) or (46) . Cohomologies will be seen to preserve the ergodic invariants of a linear random dynamical system, the Lyapunov exponents. The Oseledets spaces of the two conjugate systems will be seen to naturally correspond to each other via the cohomology mapping. Equipped with this observation, the game to be played is rather easy: the local linearization result readily extends from random to stochastic differential equations, as will be seen in the following section.

The local linearization for random differential equations we deal with in this section has been treated in Wanner [18]. The proof we shall give is much shorter than Wanner's. In addition, we will not need any assumptions concerning the block diagonal form of the linear part of the random differential equation considered. The normal form theory of [4] and [2] attempts a globalized version of linearization with $C^\infty-$transformations instead of just homeomorphic ones. Hyperbolicity is replaced by more restrictive *non-resonance* conditions.

If the Lipschitz constants for the deviations from linearity $x \mapsto f(x) - Ax$ are small enough, then in the deterministic setting we can obtain a global linearization result with which we will start our analysis. We shall modify the essential parts of the arguments used by Palmer [16] to suit the needs of random systems, in particular so that it fits well in the framework of Oseledets' theorem. It will turn out to be practical in the proof to employ the following simple topological argument (see also Protter [17]).

Proposition 3.1 *Let $h : \mathbf{R}^d \to \mathbf{R}^d$ be continuous and one-to-one. Suppose further that $\lim_{|x|\to\infty} |h(x)| = \infty$. Then h is a homeomorphism of $\mathbf{R}^d$.*

Proof:
Let $T = \mathbf{R}^d \cup \{\infty\}$ denote the Alexandrov compactification of $\mathbf{R}^d$. Extend h to T by setting

$$k : T \to T, x \mapsto \begin{cases} h(x), & \text{if} \quad x \in \mathbf{R}^d, \\ \infty, & \text{if} \quad x = \infty. \end{cases}$$

T being compact, k is a homeomorphism onto its range. Furthermore, T is homeomorphic to the sphere S^d. It is well known that S^d is not homeomorphic to nontrivial subsets of itself. Hence k must be a homeomorphism of T, and consequently h a homeomorphism of $\mathbf{R}^d$. □

Let us now turn to the local linearization of deterministic systems. To fit with our non-autonomous setting in the random differential equation case, we shall consider non-autonomous deterministic equations. Hyperbolicity of the corresponding matrix valued function $A(t), t \in \mathbf{R}$, at $t = 0$ will appear in a form in which it can be best exploited in the context of multiplicative ergodic theory later. The non-existence of vanishing real parts of eigenvalues of $A(0)$ will be hidden in the existence of two (not necessarily orthogonal) projectors P^+ and P^- on the vector product of the stable eigenspaces, i.e. those belonging to negative eigenvalues, and the product of the unstable eigenspaces, so that $P^+ + P^- = \mathrm{id}_{\mathbf{R}^d}$. The constant α appearing in the following Proposition will play the role of a lower bound on the smaller of the two spectral gaps between 0 and the positive and negative eigenvalues of $A(0)$.

Proposition 3.2 *Let $\alpha > 2c > 0$, $A : \mathbf{R} \to \mathbf{R}^{d\times d}$ be continuous, and Φ the flow of diffeomorphisms generated by the linear differential equation*

$$dy_t = A(t)\, y_t\, dt, \quad t \in \mathbf{R}. \tag{47}$$

Suppose that there exist linear projectors P^+, P^- on $\mathbf{R}^d$, $0 < \epsilon < \alpha$ and a function $R_\epsilon : \mathbf{R} \to]0, \infty[$ such that

$$\begin{aligned} ||\Phi_t P^+ \Phi_s^{-1}|| &\leq R_\epsilon(t) e^{-\alpha|t-s|} \quad \textit{for } s \geq t, \\ ||\Phi_t P^- \Phi_s^{-1}|| &\leq R_\epsilon(t) e^{-\alpha|t-s|} \quad \textit{for } s \leq t, \\ R_\epsilon(s+t) &\leq e^{\epsilon|t|} R_\epsilon(s), \quad \textit{for } s, t \in \mathbf{R}. \end{aligned}$$

Let $f : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ be bounded and continuous such that $f(\cdot, 0) = 0$. For $t \in \mathbf{R}$ let $f(t, \cdot)$ be Lipschitz continous with Lipschitz constant $L(t)$, and suppose

$$L(t)\, R_\epsilon(t) \leq c.$$

Let Ψ be the flow generated by the non-linear differential equation

$$dx_t = [A(t)\, x_t + f(t, x_t)]\, dt. \tag{48}$$

Then there exists a continuous mapping $h : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying

(i) $h(t, \cdot)$ is a homeomorphism of $\mathbf{R}^d$ and $h(t, 0) = 0, \quad t \in \mathbf{R}$,

(ii) $\Phi_t = h(t, \cdot) \circ \Psi_t \circ h(0, \cdot)^{-1}, \quad t \in \mathbf{R}$.

Proof:
For $t \in \mathbf{R}$, $x \in \mathbf{R}^d$, define

$$h(t,x) = x + \int_t^\infty \Phi_t P^+ \Phi_s^{-1}\ f(s, \Psi_s \Psi_t^{-1} x)\, ds - \int_{-\infty}^t \Phi_t P^- \Phi_s^{-1}\ f(s, \Psi_s \Psi_t^{-1} x)\, ds.$$

Due to the inequalities of the hypothesis, h is well defined and continuous, and we have $h(t, 0) = 0$. More precisely, we may estimate

$$|h(t,x) - x| \leq R_\epsilon(t)\, ||f||_\infty\, [\int_t^\infty e^{-\alpha|t-s|} ds + \int_{-\infty}^t e^{-\alpha|t-s|} ds] = \frac{2R_\epsilon(t)\, ||f||_\infty}{\alpha}. \tag{49}$$

Let us first prove that h maps solutions $(x_t)_{t\in\mathbf{R}}$ of (48) to solutions of (47). Indeed, for $t \in \mathbf{R}$, let $y_t = h(t, x_t)$. Since $\Psi_s \Psi_t^{-1}\, x_t = \Psi_s\, x_0 = x_s$, we obtain

$$y_t = h(t, x_t) = x_t + \Phi_t\, [\int_t^\infty P^+ \Phi_s^{-1}\ f(s, x_s)\, ds - \int_{-\infty}^t P^- \Phi_s^{-1}\ f(s, x_s)\, ds],$$

hence by differentiating

$$\begin{aligned} \frac{d}{dt} y_t &= \frac{d}{dt} x_t + \frac{d}{dt} \Phi_t\, [\int_t^\infty P^+ \Phi_s^{-1}\ f(s, x_s)\, ds - \int_{-\infty}^t P^- \Phi_s^{-1}\ f(s, x_s)\, ds] \\ &\quad + \Phi_t[-P^+ \Phi_t^{-1}\, f(t, x_t) - P^- \Phi_t^{-1}\, f(t, x_t)] \\ &= A(t)\, x_t + f(t, x_t) \\ &\quad + A(t)\, \Phi_t[\int_t^\infty P^+ \Phi_s^{-1}\ f(s, x_s)\, ds - \int_{-\infty}^t P^- \Phi_s^{-1}\ f(s, x_s)\, ds] \\ &\quad - f(t, x_t) \\ &= A(t)\, h(t, x_t) = A(t)\, y_t. \end{aligned}$$

This also proves the cohomology property

$$\Phi_t = h(t,\cdot) \circ \Psi_t \circ h(0,\cdot)^{-1}, \quad t \in \mathbf{R}.$$

This property reveals that in order to prove $h(t,\cdot)$ to be a homeomorphism for all $t \in \mathbf{R}$, it is enough to show this for $t = 0$. So it remains to show that $h(0,\cdot)$ is a homeomorphism. But (48) implies that $\lim_{|x|\to\infty} |h(0,x)| = \infty$. Hence Proposition 3.1 tells us that it is enough to establish that $h(0,\cdot)$ is one-to-one. For this purpose, let $x, y \in \mathbf{R}^d$ be such that $h(0,x) = h(0,y)$. Let ξ resp. η be solutions of (48) with initial conditions x resp. y, and set $\rho = \xi - \eta$. Then, writing $b(t) = f(t,\eta_t) - f(t,\xi_t)$, $t \in \mathbf{R}$, ρ satisfies the differential equation

$$\frac{d}{dt}\rho_t = A(t)\,\rho_t + b(t),$$

whose solution, by variation of constants, is seen to satisfy the equations

$$\rho_t = \Phi_t\,[\Phi_{t_0}^{-1}\,\rho_{t_0} + \int_{t_0}^{t} \Phi_s^{-1}\, b(s)ds], \quad t_0, t \in \mathbf{R}. \tag{50}$$

We will establish $\rho = 0$. To do this, let us first estimate the unstable part of (50) in the form

$$\begin{aligned} |\Phi_t P^- \Phi_t^{-1} \rho_t| &= |\Phi_t\, P^- \Phi_{t_0}^{-1}\, \rho_{t_0} + \int_{t_0}^{t} \Phi_t P^- \Phi_s^{-1}\, b(s)ds| \\ &\leq ||\Phi_t\, P^- \Phi_{t_0}^{-1}||\, |\rho_{t_0}| + \int_{t_0}^{t} ||\Phi_t P^- \Phi_s^{-1}||\, |b(s)|ds \\ &\leq R_\epsilon(t) e^{-\alpha|t-t_0|} |\rho_{t_0}| + R_\epsilon(t) \int_{t_0}^{t} e^{-\alpha|t-s|}\, L(s)|\rho_s|\, ds, \end{aligned} \tag{51}$$

$t_0 \leq t$. For $|\rho_{t_0}|$ in the last line of (51) a less accurate estimate will be sufficient. To get this estimate, note that by uniqueness of solutions for (47) we have $h(t,\xi_t) = h(t,\eta_t), t \in \mathbf{R}$. Hence for $t \in \mathbf{R}$ by (49)

$$|\rho_t| \leq |\xi_t - h(t,\xi_t)| + |\eta_t - h(t,\eta_t)| \leq \frac{4R_\epsilon(t)\, ||f||_\infty}{\alpha}. \tag{52}$$

Substituting (52) into (51) yields the estimate

$$|\Phi_t P^- \Phi_t^{-1} \rho_t| \leq R_\epsilon(t) e^{-\alpha|t-t_0|} \frac{4R_\epsilon(t_0)\, ||f||_\infty}{\alpha} + R_\epsilon(t) \int_{t_0}^{t} e^{-\alpha|t-s|}\, L(s)R_\epsilon(s)\, ds|| \frac{\rho}{R_\epsilon}||_\infty. \tag{53}$$

Now with $t_0 \to -\infty$, using the hypothesis $\epsilon < \alpha$ and the bound c for the product $L(s)R_\epsilon(s)$, we get

$$\frac{|\Phi_t P^- \Phi_t^{-1} \rho_t|}{R_\epsilon(t)} \leq \frac{c}{\alpha}||\frac{\rho}{R_\epsilon}||_\infty. \tag{54}$$

We finally estimate the stable part of ρ in an analogous way to get the inequality

$$||\frac{\rho}{R_\epsilon}||_\infty \leq \frac{2c}{\alpha}||\frac{\rho}{R_\epsilon}||_\infty.$$

Since $2c < \alpha$, this is only possible if $\rho = 0$. This completes the proof. $\Box$

For the next step of our program we now return to the stochastic setting. Our goal is to generalize the preceding result to the setting of random differential equations. The stochastic analogues of the linear projectors $P^\pm$ are given by the random projectors on the stable and unstable parts of the Oseledets spectrum. We shall recall some well known facts about multiplicative ergodic theory of linear random cocycles. For more details see Arnold [1].

Proposition 3.3 *Let $(\Omega, \mathcal{F}, P)$ together with a group $(\theta_t)_{t\in\mathbf{R}}$ of $P-$preserving transformations of Ω be an ergodic metric dynamical system, Φ a linear cocycle with this basis, such that the integrability condition of the MET*

$$\sup_{0\le t\le 1} [\ln^+ ||\Phi_t|| + \ln^+ ||\Phi_t^{-1}||] \quad \textit{is integrable}$$

holds. Let $\lambda_1, \cdots, \lambda_p$ be the Lyapunov exponents, suppose that none of them vanishes, i.e. Φ is hyperbolic. Let $E_1, \cdots, E_p$ be the corresponding Oseledets spaces. Moreover, let

$$E^+ = \oplus_{\lambda_i>0} E_i, \quad E^- = \oplus_{\lambda_i<0} E_i,$$

the splitting of $\mathbf{R}^d$ into unstable and stable parts, P^+, P^- the associated random linear projectors.

Finally, let $\alpha < \min_{1\le i\le p} |\lambda_i|$. Then there exists $0 < \epsilon < \alpha$ and a random variable $R_\epsilon : \Omega \to [1, \infty[$ satisfying

*(i) R_ϵ is $\epsilon-$*slowly varying, *i.e. for $t \in \mathbf{R}$*

$$R_\epsilon(\theta_t\cdot) \le e^{\epsilon|t|} R_\epsilon,$$

(ii)for $s, t \in \mathbf{R}$ we have

$$\begin{aligned} ||\Phi_t P^+ \Phi_s^{-1}|| &\le R_\epsilon(\theta_t\cdot) e^{-\alpha|t-s|} \quad \textit{for } s \ge t, \\ ||\Phi_t P^- \Phi_s^{-1}|| &\le R_\epsilon(\theta_t\cdot) e^{-\alpha|t-s|} \quad \textit{for } s \le t. \end{aligned}$$

Proof:
This is a consequence of Theorem 4.3.4 and Corollary 4.3.11 of Arnold [1], where a variant of the statement based on the single Oseledets spaces is proved. Its generalization to the whole unstable or stable part of the Oseledets spectrum is straightforward. To better adapt the arguments given in Arnold [1], note that the cocycle property and the invariance of the Oseledets spaces implies for $s, t \in \mathbf{R}$

$$\Phi_t P^\pm \Phi_s^{-1} = \Phi_t P^\pm(\theta_{-s}\theta_s\cdot)\Phi_{-s}(\theta_s\cdot) = \Phi_t \Phi_{-s}(\theta_s\cdot) P^\pm(\theta_s\cdot) = \Phi_{t-s}(\theta_s\cdot) P^\pm(\theta_s\cdot).$$

$\Box$

We are ready to prove a global linearization result for random differential equations. Given the random vector field f and its linearization A at the fixed point 0, we will have to impose continuity conditions concerning the maps $t \mapsto A(\theta_t\omega)$ as well as $(t,x) \mapsto f(\theta_t\omega, x)$, which may seem restrictive at first glance. We therefore recall that the linearization results for rde are of auxiliary character for our ultimate aims: in the following section on local linearization of sde, these vector fields will be naturally derived from the vector fields determining the sde and their linearizations in 0. The continuity conditions will then be seen to be easy consequences of the smoothness properties of the cohomology mediating the passage between sde and rde.

Theorem 3.1 *Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ be an ergodic metric dynamical system, $A : \Omega \to \mathbf{R}^d \times \mathbf{R}^d$ a random matrix, and $f : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ a random vector field such that $f(\cdot, 0) = 0$. Suppose that for all $\omega \in \Omega$ the maps $t \mapsto A(\theta_t\omega)$ as well as $(t,x) \mapsto f(\theta_t\omega, x)$ are continuous.*

Let Φ be the flow of the random differential equation

$$dy_t = A(\theta_t\cdot)\, y_t\, dt, \quad t \in \mathbf{R}, \tag{55}$$

and suppose that Φ satisfies the integrability property of the MET and is hyperbolic. Let $P^+, P^-, \alpha, \epsilon$, and R_ϵ be given according to Proposition 3.3. For $\omega \in \Omega$, let $f(\omega, \cdot)$ be Lipschitz continuous with (measurable) Lipschitz constant $L(\omega)$, and let a constant $c < \frac{\alpha}{2}$ be given such that

$$L\, R_\epsilon \leq c.$$

Let Ψ be the random cocycle generated by the random differential equation

$$dx_t = [A(\theta_t\cdot)\, x_t + f(\theta_t\cdot, x_t)]\, dt. \tag{56}$$

Then there exists a measurable mapping $h : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying the following properties

(i) $h(\omega, \cdot)$ is a homeomorphism of $\mathbf{R}^d$ and $h(\omega, 0) = 0, \quad \omega \in \Omega$,

(ii) $\Phi_t = h(\theta_t\cdot, \cdot) \circ \Psi_t \circ h(\cdot, \cdot)^{-1}, \quad t \in \mathbf{R}$.

Proof:
Apply Proposition 3.2 ω-by-ω. This will yield a measurable mapping $h : \Omega \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ such that

(i) $h(\omega, t, \cdot)$ is a homeomorphism of $\mathbf{R}^d$ and $h(\omega, t, 0) = 0, \quad \omega \in \Omega$,

(ii) $\Phi_t = h(\cdot, t, \cdot) \circ \Psi_t \circ h(\cdot, 0, \cdot)^{-1}, \quad t \in \mathbf{R}$.

The explicit integral representation given in the proof of Proposition 3.2 and the cocycle properties of Φ and Ψ then yield for $t \in \mathbf{R}$

$$h(\cdot, t, \cdot) = h(\theta_t\cdot, 0, \cdot).$$

We therefore have to take $h(\omega, \cdot) = h(\omega, 0, \cdot), \omega \in \Omega$. $\square$

The existence and at the same time smallness of a random Lipschitz constant required in the global linearization theorem is of course far from being guaranteed in general. Given a non-linear random differential equation, we will have to modify the non-linear parts of the vector fields by cutting them where they become too big. This leads to the following local linearization theorem which is much more natural for stochastic systems.

Theorem 3.2 *Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ be an ergodic metric dynamical system, $F : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ a random vector field satisfying the following properties*

(i) for $\omega \in \Omega$ we have $F(\omega, \cdot) \in \mathcal{C}^1(\mathbf{R}^d), \quad F(\omega, 0) = 0$,

(ii) for $\omega \in \Omega$ the maps $(t, x) \mapsto F(\theta_t\omega, x)$, and $(t, x) \mapsto \frac{\partial}{\partial x} F(\theta_t\omega, x)$ are continuous.

Let $A = \frac{\partial}{\partial x} F(\cdot, 0)$, let Φ be the flow generated by the random differential equation

$$dy_t = A(\theta_t \cdot)\, y_t\, dt, \quad t \in \mathbf{R}, \tag{57}$$

and suppose that Φ satisfies the integrability property of the MET and is hyperbolic.

Let Ψ be the local random cocycle generated by the random differential equation

$$dx_t = F(\theta_t \cdot, x_t)\, dt. \tag{58}$$

Then there exist measurable mappings $\rho : \Omega \to]0, \infty[$ and $h : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying the following properties

(i) $h(\omega, \cdot)$ is a homeomorphism of $\mathbf{R}^d$ and $h(\omega, 0) = 0, \quad \omega \in \Omega$,

(ii) $t \mapsto \rho(\theta_t\omega)$ is continuous, $\omega \in \Omega$,

(iii) for $\omega \in \Omega, x \in \mathbf{R}^d, \tau_-(\omega, x) \le t \le \tau_+(\omega, x)$ we have $\Phi_t(\omega) = h(\theta_t\omega, \cdot) \circ \Psi_t(\omega) \circ h(\omega, x)^{-1}$, $t \in \mathbf{R}$, where

$$\begin{aligned}
\tau_-(\omega, x) &= \inf\{t < 0 : |\Psi_s(\omega, x)| \le \rho(\theta_s\omega) \text{ for all } t \le s \le 0\}, \\
\tau_+(\omega, x) &= \sup\{t > 0 : |\Psi_s(\omega, x)| \le \rho(\theta_s\omega) \text{ for all } 0 \le s \le t\}.
\end{aligned}$$

Proof:
Let

$$f(\cdot, x) = F(\cdot, x) - A\, x, \quad x \in \mathbf{R}^d,$$

be the non-linear part of F. The problem we face being a local one, we may and do assume that $f(\omega, \cdot)$ vanishes outside a deterministic compact set for all $\omega \in \Omega$. So in particular Ψ may be assumed global. Our hypotheses allow to apply Proposition 3.3 to the random dynamical system Φ. So let $\alpha, 0 < \epsilon < \alpha$, and R_ϵ be given according to Proposition 3.3.

Let $0 < c < \frac{\alpha}{2}$ be arbitrary. To be able to apply the global linearization result of the preceding Theorem, we have to find suitable random neighborhoods of time 0 and the origin in $\mathbf{R}^d$ so that the product of R_ϵ with the Lipschitz constant of the generator of Ψ does not exceed the bound c there, and then modify the generator so that this is valid globally.

To this end, define the random Lipschitz bound for the non-linear part on balls of radius r by

$$\Lambda(\omega, r) = \sup_{|x|\leq r} |\frac{\partial}{\partial x} f(\omega, x)| + r, \quad \omega \in \Omega, r \geq 0,$$

and determine the random radius at which the bound c is reached by ΛR_ϵ via the equation

$$\rho(\omega) = \sup\{r > 0 : \Lambda(\omega, r)\, R_\epsilon(\omega) \leq c\}.$$

Note that $\Lambda(\cdot, 0) = 0$, so that ρ is indeed positive. The hypotheses on F moreover yield that the random mapping $(t, r) \mapsto \Lambda(\theta_t \cdot, r)$ is pointwise continuous and strictly increasing in r. By definition of $\epsilon-$slow variation also $t \mapsto R_\epsilon(\theta_t \cdot)$ is pointwise continuous. This implies that also $t \mapsto \rho(\theta_t \omega)$ is continuous. So (ii) of the assertion is verified for ρ.

Let us next modify the non-linear part of the generator on the random neighborhoods to obtain a generator with small enough Lipschitz bounds. Let

$$g(\cdot, x) = f(\cdot, \chi_\rho(x)),$$

where for $r > 0$ we use the retraction mapping from $\mathbf{R}^d$ to the ball of radius r defined by

$$\chi_r(x) = x\, 1_{[0,\rho]}(|x|) + \frac{x}{|x|}\, 1_{]\rho,\infty[}(|x|).$$

The retraction mapping being Lipschitz with Lipschitz constant 1, we conclude that for $\omega \in \Omega$, $g(\omega, \cdot)$ is globally Lipschitz with a Lipschitz constant $L(\omega)$ satisfying the required

$$L\, R_\epsilon \leq c.$$

We may now apply Theorem 3.1 to the random cocycle Ξ generated by the random differential equation

$$dx_t = [A(\theta_t \cdot)\, x_t + g(\theta_t \cdot, x_t)]\, dt, \tag{59}$$

to obtain a random coordinate transform h satisfying (i) of the asserted properties. Since the trajectories of Ψ and Ξ coincide on the random intervals $[\tau_-, \tau_+]$, (iii) follows. Finally, note that continuity implies $\inf_{s\leq t\leq u} \rho(\theta_t \cdot) > 0$ for any interval $[s, u] \subset \mathbf{R}$. This implies that τ_-, τ_+ are positive which says that the assertion in (iii) is not trivial. The proof is complete. □

4 Local linearization of stochastic differential equations

We now come to the final step of our program. Given a random dynamical system generated by a stochastic differential equation, we want to relate it by local homeomorphisms to its linearization in the fixed point 0. To do this, we shall pass to a conjugate random differential equation. For the cocycle of the latter and its linearization in the fixed point 0, the local linearization results of the previous section will be applied. Passing back to the sde setting via the cohomology, we will find the required homeomorphism relating locally the cocycles of the nonlinear sde and its linearization in the fixed point 0. Of course, the passage from sde to rde and back via stationary coordinate changes has to preserve concepts of multiplicative ergodic theory, such as Lyapunov exponents and Oseledets spectra. The following Proposition states that this is in fact the case, if the coordinate change satisfies some basic integrability requirements.

Proposition 4.1 *Let Φ be a linear cocycle which fulfills the integrability of the MET, i.e.*

$$\sup_{0\le t\le 1}[\ln^+ ||\Phi_t|| + \ln^+ ||\Phi_t^{-1}||] \quad \textit{is integrable.}$$

Let moreover $H : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ be a random linear mapping satisfying

(i) $t \mapsto H(\theta_t \cdot)$ is continuous, $\omega \in \Omega$,

(ii) H satisfies the integrability condition

$$\sup_{0\le t\le 1}[\ln^+ ||H(\theta_t \cdot)|| + \ln^+ ||H(\theta_t \cdot)^{-1}||] \quad \textit{is integrable.} \tag{60}$$

Then

$$\Psi_t = H(\theta_t \cdot)\,\Phi_t\,H^{-1}, \quad t \in \mathbf{R}, \tag{61}$$

defines a linear cocycle which satisfies the integrability condition of the MET. Ψ possesses the same Lyapunov exponents as Φ. If $E_1, \cdots, E_p$ are the Oseledets spaces of Φ, then the Oseledets spaces of Ψ are given by $H\,E_1, \cdots, H\,E_p$.

Proof:
The cocycle and integrability properties for Ψ are immediate from our hypotheses. Stationarity and a simple Borel-Cantelli argument moreover show that

$$\lim_{t\to\pm\infty} \frac{1}{t} \ln^+ ||H(\theta_t \cdot)|| = 0 \quad P-\text{a.s.}.$$

This easily entails that Φ and Ψ have identical Lyapunov exponents, while the Oseledets spaces are related in the asserted way. □

Remark:
In the terminology of Arnold [1], Proposition 4.1 contains the statement that H is *tempered* if it satisfies the integrability condition (60).

We can now state the main result of this section. We shall impose additional integrability assumptions for the vector fields of our sde. They are needed for the following reasons. The passage from the sde to a cohomologous rde requires the existence of a stationary diffeomorphism according to Theorem 2.2. The Jacobian of this diffeomorphism enters into the random vector field g generating the rde. In a second step, this vector field has to be linearized. Hence, we obviously need the stationary diffeomorphism to be C^2. This fact formally increases the smoothness degree of the vector fields by one compared to the theory exposed in section 2. Now note that the passage to cohomologous rde is just an auxiliary step in our algorithm which primarily aims at obtaining a homeomorphism for the local linearization. So it is natural to suggest that the additional smoothness required formally can probably be disposed of. One possibility to justify this conjecture would consist in first smoothing the vector fields uniformly e.g. by a smooth convolution, obtaining the associated homeomorphism, and then going to the limit on the homeomorphism level, without explicit intervention of cohomologous rde.

Theorem 4.1 *Let $\delta > 0$. Suppose that $f_0 \in \mathcal{C}_b^{2,\delta}$, $f_1, \cdots, f_m \in \mathcal{C}_b^{3,\delta}$, and $\sum_{i=1}^m \sum_{j=1}^d f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{3,\delta}$. Suppose further that 0 is a fixed point for the vector fields $f_0, \cdots, f_m$, and let $A_i = \frac{\partial}{\partial x} f_i(0), 0 \leq i \leq m$.*

Let Φ be the cocycle generated by the linear sde

$$dy_t = A_0\, y_t\, dt + \sum_{i=1}^m A_i\, y_t \circ dW_t^i, \quad t \in \mathbf{R}, \tag{62}$$

and suppose that Φ is hyperbolic, i.e. all Lyapunov exponents are non-zero. Let Ψ be the cocycle generated by the sde

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i, \quad t \in \mathbf{R}. \tag{63}$$

Then there exist measurable mappings $\rho : \Omega \to]0, \infty[$ and $h : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying the following properties

(i) $h(\omega, \cdot)$ is a homeomorphism of $\mathbf{R}^d$ and $h(\omega, 0) = 0, \quad \omega \in \Omega$,

(ii) $t \mapsto \rho(\theta_t \omega)$ is continuous, $\omega \in \Omega$,

(iii) for $\omega \in \Omega, x \in \mathbf{R}^d, \tau_-(\omega, x) \leq t \leq \tau_+(\omega, x)$ we have $\Phi_t(\omega) = h(\theta_t \omega, \cdot) \circ \Psi_t(\omega) \circ h(\omega, x)^{-1}, \quad t \in \mathbf{R}$, where

$$\begin{aligned} \tau_-(\omega, x) &= \inf\{t < 0 : |\Psi_s(\omega, x)| \leq \rho(\theta_s \omega) \text{ for all } t \leq s \leq 0\}, \\ \tau_+(\omega, x) &= \sup\{t > 0 : |\Psi_s(\omega, x)| \leq \rho(\theta_s \omega) \text{ for all } 0 \leq s \leq t\}. \end{aligned}$$

Proof:
Choose H and Γ according to Theorem 2.2, let

$$g(\cdot, y) = \frac{\partial}{\partial x} H_0^{-1}(y)[f_0(H_0(y)) + \Gamma_0(y)], \quad y \in \mathbf{R}^d,$$

and denote by Ψ^0 the cocycle associated with the random differential equation induced by g. Since 0 is a fixed point for all the vector fields involved in (63), by construction this is the case for H_0 and Γ_0. Hence also $g(\cdot, 0) = 0$. By our differentiability hypotheses we know that the vector field g may be differentiated at 0. Let

$$A = \frac{\partial}{\partial x} g.$$

By construction of H and Γ we further know that the flow Φ^0 of the linearized random differential equation

$$dy_t = A(\theta_t \cdot)\, y_t\, dt, \quad t \in \mathbf{R}, \tag{64}$$

is conjugate to Φ via the cohomology $\frac{\partial}{\partial x} H$. We now have to show that Theorem 3.2 is applicable to the cocycles Φ^0 and Ψ^0.

By Theorems 2.1 and 2.2, g satisfies the smoothness properties required in Theorem 3.2. It remains to see that Φ^0 satisfies the integrability properties of the MET and is hyperbolic. Both conditions follow from Proposition 4.1 provided we can establish that Φ satisfies the integrability condition of the MET, and

$$\sup_{0 \le t \le 1} [\ln^+ ||H_0(\theta_t \cdot)|| + \ln^+ ||H_0(\theta_t \cdot)^{-1}||] \quad \text{is integrable.} \tag{65}$$

The linear cocycle Φ automatically satisfies the integrability conditions of the MET. Property (65) follows from the construction of H and the boundedness properties of moments of the flows of diffeomorphisms Φ^α established in the proof of Proposition 2.1 which led to the definition of H in Theorem 2.1.

Now we apply Theorem 3.2, and denote by h^0 the random homeomorphism and by ρ^0 the random radius of neighborhoods of 0 it provides. We finally have to set

$$h = \frac{\partial}{\partial x} H_0(\cdot, 0) \circ h^0 \circ H_0^{-1},$$

and modify ρ^0 slightly by multiplying it with random constants depending on the positive random variable $||\frac{\partial}{\partial x} H_0||$. This completes the proof. □

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