Two Mathematical Approaches to Stochastic Resonance

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Summary. We consider a random dynamical system describing the diffusion of a small-noise Brownian particle in a double-well potential with a periodic perturbation of very large period. According to the physics literature, the system is in stochastic resonance if its random trajectories are tuned in an optimal way to the deterministic periodic forcing. The quality of periodic tuning is measured mostly by the amplitudes of the spectral components of the random trajectories corresponding to the forcing frequency. Reduction of the diffusion dynamics in the small noise limit to a Markov chain jumping between its meta-stable states plays an important role.

We study two different measures of tuning quality for stochastic resonance, with special emphasis on their robustness properties when passing to the reduced dynamics of the Markov chains in the small noise limit. The first one is the physicists favourite, spectral power amplification. It is analyzed by means of the spectral properties of the diffusion's infinitesimal generator in a framework where the system switches every half period between two spatially antisymmetric potential states. Surprisingly, resonance properties of diffusion and Markov chain differ due to the crucial significance of small intra-well fluctuations for spectral concepts. To avoid this defect, we design a second measure of tuning quality which is based on the pure transition mechanism between the meta-stable states. It is investigated by refined large deviation methods in the more general framework of smooth periodically varying potentials, and proves to be robust for the passage to the reduced dynamics.

1 Introduction: model reduction and resonance

On an appropriate probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider a one-dimensional diffusion $X^{\varepsilon,T} = (X_t^{\varepsilon,T})_{t\geq 0}$ driven by the stochastic differential equation

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{T})dt + \sqrt{\varepsilon}dW_t, \quad X_0^{\varepsilon,T} = x \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

where W is a standard Brownian motion, and ε a small noise intensity. The potential U is supposed to be double-well in the spatial coordinate. It is temporally periodic with period 1, i.e. $U(\cdot, t) = U(\cdot, t+1)$, for any $t \geq 0$. U' is used to denote the derivative in x. The positive parameter T stand for the period of the deterministic perturbation. As an example for U one can take $U(x,t) = U_0(x) + ax \sin 2\pi t$, $x \in \mathbb{R}$, $t \geq 0$, with a symmetric potential $U_0 = \frac{x^4}{4} - \frac{x^2}{2}$, $x \in \mathbb{R}$, and a small enough amplitude a to ensure that U does not degenerate to a one-well potential.

Given a large period T, we will be interested in *periodicity properties* of the trajectories of our system (1), in particular to have a mathematically precise concept of how well they are able to follow the deterministic periodic excitation in dependence on the noise intensity ε. It is intuitively clear that if ε is very small, trajectories will almost never be able to leave the well in which they start, and stay close to the starting well's minimum (see Fig.1 (l.)). If ε is very large, the energy of the particle is sufficient to trigger some chaotic changing between the two wells. There will be an intermediate range of small intensities, at which trajectories are more or less close to the deterministic periodic function describing the temporally varying energetically most favorable position in the potential landscape given by the minimum of the deeper of the two wells (see Fig.1 (r.), [Fre00]). The crucial questions to be answered in the sequel are the following. Given T large, for which intensity $\varepsilon(T)$ will the periodicity of the system's trajectories be optimal? And how can we measure the quality of periodicity?

The quasi-deterministic behaviour the trajectories exhibit for small noise raises another question which is very significant since it indicates a route to reduction of complexity relevant for high dimensional systems: is it possible to study the periodic tuning properties of the diffusion by considering instead a simplified two-state Markov chain model, which catches the diffusion's *effective dynamics*?

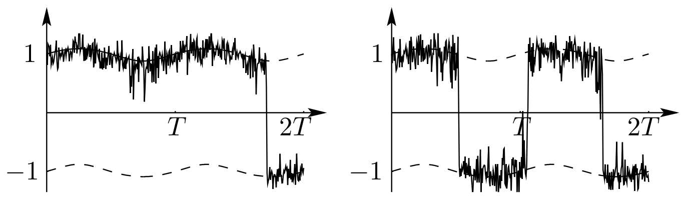

Fig. 1. Sample paths of (1) for small (l.) and 'optimal' (r.) values of ε

This approach was extensively studied by physicists. In their pioneering theoretical paper [MW89] McNamara and Wiesenfeld propose a two-state model of stochastic resonance in which the small-noise diffusion in a double-well potential *in adiabatic limit* is replaced by a two-state Markov process.

Along with (1) the inhomogeneous Markov chain $Y^{\varepsilon,T}$ is considered, which possesses the 1-periodic infinitesimal generator

$$Q_{\varepsilon,T}(t) = \begin{pmatrix} -\varphi(\frac{t}{T}) & \varphi(\frac{t}{T}) \\ \psi(\frac{t}{T}) & -\psi(\frac{t}{T}) \end{pmatrix} \tag{2}$$

with the infinitesimal transition rates φ and ψ given by time-perturbed *Kramers-Eyring law* [Kra40]

$$\begin{aligned} \varphi(t) &= \frac{1}{2\pi}\sqrt{|U_0''(0)|U_0''(1)}e^{-\frac{2}{\varepsilon}(\Delta U + a\sin 2\pi t)}, \\ \psi(t) &= \frac{1}{2\pi}\sqrt{|U_0''(0)|U_0''(1)}e^{-\frac{2}{\varepsilon}(\Delta U - a\sin 2\pi t)}, \quad t \geq 0, \end{aligned} \tag{3}$$

where $\Delta U = \frac{1}{4}$ is the height of the potential barrier of the unperturbed potential U_0. Kramers obtained his law heuristically in the autonomous case, i.e. for $a = 0$.

To measure periodicity of the random trajectories of either the diffusion or the Markov chain, we first take the so-called coefficient of *spectral power amplification* (SPA), one of the physicists' favourite characteristics, see e.g. [BPSV83, MW89, DLM+95, GHJM98, ANMS99, WJ98]. It is based on the power spectrum of the average trajectories with respect to the equilibrium of the homogenized Markov processes $(X^{\varepsilon,T}_{Tt}, t \pmod 1)_{t\geq 0}$ resp. $(Y^{\varepsilon,T}_{Tt}, t \pmod 1)_{t\geq 0}$. For the diffusion (1) with equilibrium μ it is defined by

$$\eta^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu(X^{\varepsilon,T}_{Ts}) \cdot e^{2\pi i s}\, ds \right|^2. \tag{4}$$

The function η^X depending on noise intensity and the period of time variation of the potential has a clear physical meaning. It describes the amount of energy carried by the averaged path of the diffusion with noise amplitude ε on the frequency $\frac{2\pi}{T}$. The expectation $\mathbf{E}_\mu$ indicates that averages are taken with respect to μ. This will be explained in detail later.

Fig. 2 borrowed from [ANMS99] where Ω corresponds to our $\frac{2\pi}{T}$ and D to the diffusion intensity ε shows that physicists expect a local maximum of the function $\varepsilon \mapsto \eta^X(\varepsilon, \cdot)$. The random paths have their strongest periodic component corresponding to the frequency of the periodic input at the value of ε for which the maximum is taken. In fact, Fig. 2 does not show the SPA coefficient of the diffusion itself, but of its *effective dynamics* given by the two-state Markov chain $Y^{\varepsilon,T}$. It is *a priori* believed in the physical literature that the effective dynamics adequately describes the properties of the diffusion in the small noise limit.

If periodic tuning is measured by SPA, to determine the 'optimal tuning' or *stochastic resonance point* means to find the argument $\varepsilon = \varepsilon(T)$ of a local maximum of $\varepsilon \mapsto \eta^X(\varepsilon, \cdot)$ for large T. In section 2 we address the problem of finding the stochastic resonance point for the diffusion by means of the

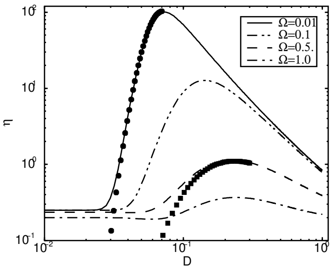

Fig. 2. SPA coefficient as a function of noise amplitude is supposed to have a well pronounced maximum depending on the frequency of periodic perturbation [ANMS99].

passage to its effective dynamics in the small noise limit. We shall see that determining the optimal tuning intensity $\varepsilon(T)$ for the Markov chain is a relatively easy task. It turns out, however, that already in the very simple case of a potential hopping every half period between two spatially antisymmetric double-well states with wells of different depths, due to the crucial importance of the diffusion's *inter-well* fluctuations, i.e. small fluctuations in the vicinity of the potential's minima, at low noise, diffusion and Markov chain exhibit different resonance features. In contradiction to physicists' intuition, the SPA notion of resonance is therefore not robust when passing to the reduced model. In section 3 we therefore propose a concept of measuring the quality of periodic tuning which is based on the pure mechanism of transition between the domains of attraction of the potential's local minima, and therefore fails to have this robustness defect.

2 Periodically switching potentials and the spectral approach

To catch the essentials of the effect and at the same time to simplify the problem we will work in the first part of this paper with a time-space antisymmetric double well potential switching discontinuously between two states. In the second part we will essentially extend this framework to include continuously varying potentials. In the strip $(x,t) \in \mathbb{R} \times [0,1)$ it is defined by the formula

$$U(x,t) = \begin{cases} U_1(x), & t \in [0, \frac{1}{2}), \\ U_2(x) = U_1(-x), & t \in [\frac{1}{2}, 1). \end{cases} \tag{5}$$

It is periodically extended for all times t by the relation $U(\cdot,t) = U(\cdot,t+1)$, see Fig. 3. We assume that the potential has two local minima at ± 1 and a local maximum at 0, that $U_1(-1) = -\frac{V}{2}$, $U_1(1) = -\frac{v}{2}$, $\frac{2}{3} < \frac{v}{V} < 1$, and

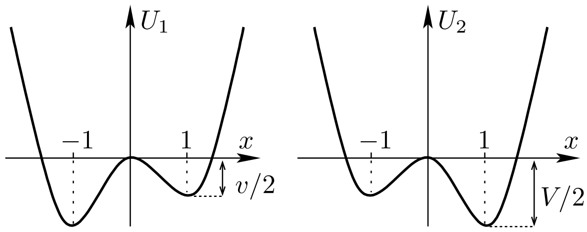

Fig. 3. Time-periodic potential U.

$U_1(0) = 0$. We also suppose that the extrema of U are not degenerate, i.e. the curvatures at these points do not vanish.

A trajectory of a Brownian particle in this potential is described by the solution of the stochastic differential equation (6).

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{T})\, dt + \sqrt{\varepsilon}\, dW_t, \quad X_0^{\varepsilon,T} = x \in \mathbb{R}. \tag{6}$$

We aim at finding a resonance intensity $\varepsilon(T)$ for large T which maximizes the SPA coefficient given by (4). The key to the solution of this problem lies in determining the time-dependent invariant density μ of $(X_{Tt}^{\varepsilon,T})_{t\geq 0}$. From now on we follow [Pav02] and [IP02]. Although the diffusion is not time homogeneous, by enlarging its state space we can consider a two-dimensional time homogeneous Markov process $(X_{Tt}^{\varepsilon,T}, t \pmod 1)$ which possesses an invariant law in the usual sense. By definition we identify the time-dependent equilibrium density μ of $(X_{Tt}^{\varepsilon,T})_{t\geq 0}$ with the invariant density of the two-dimensional process. Indeed, with respect to μ and for fixed t, the law of the real random variable $X_{Tt}^{\varepsilon,T}$ has the density $\mu(\cdot, t \pmod 1)$. The invariant density μ is a positive solution of the forward Kolmogorov (Fokker–Planck) equation $A_{\varepsilon,T}^{*}\mu = 0$, where

$$A_{\varepsilon,T}^{*}\cdot = -\frac{1}{T}\frac{\partial}{\partial t}\cdot + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\cdot + \frac{\partial}{\partial x}\left(\cdot \frac{\partial}{\partial x}U\right)$$

is the formal adjoint of the infinitesimal generator of the two-dimensional diffusion. Moreover, from the time periodicity and time-space antisymmetry of the potential U defined by (5) one concludes that $\mu(x,t) = \mu(-x, t+\frac{1}{2})$ and $\mu(x,t) = \mu(x, t+1)$, $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

This results in the following boundary-value problem used to determine μ. It is enough to solve the Fokker–Planck equation $A_{\varepsilon,T}^{*}\mu = 0$ in the strip $(x,t) \in \mathbb{R} \times [0, \frac{1}{2}]$ with boundary condition $\mu(x,0) = \mu(-x, \frac{1}{2})$, $x \in \mathbb{R}$.

2.1 The spectral gap and the first eigenfunction

We have assumed in (5) that the time dependent potential U is a step function of the time variable. In the region $(x,t) \in \mathbb{R} \times (0, \frac{1}{2})$ it is identical to a time independent double well potential U_1, and therefore the Fokker–Planck equation turns into a one-dimensional parabolic PDE

$$\frac{1}{T}\frac{\partial}{\partial t}\mu(x,t) = \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\mu(x,t) + \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial}{\partial x}U_1(x)\right). \tag{7}$$

Let L_ε^* denote the second order differential operator appearing on the right hand side of (7).

To determine μ we shall use the Fourier method of separation of variables which consists in expanding the solution of (7) into a Fourier series with respect to the system of eigenfunctions of the operator L_ε^*. It turns out that under the condition that U_1 is smooth and increases at least super-linearly at $\pm\infty$, the operator L_ε^* is essentially self-adjoint in $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U_1}{\varepsilon}}\,dx)$, its spectrum is discrete and non-positive, and the corresponding eigenspaces are one-dimensional. Denoting by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product in $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U_1}{\varepsilon}}\,dx)$ we consider the following formal Floquet type expansion

$$\mu(x,t) = \sum_{k=0}^{\infty} a_k \frac{\Psi_k(x)}{\|\Psi_k\|} e^{-T\lambda_k t}, \quad (x,t) \in \mathbb{R}\times[0,\tfrac{1}{2}], \tag{8}$$

where $\{-\lambda_k, \frac{\Psi_k}{\|\Psi_k\|}\}_{k\geq 0}$ is the orthonormal basis corresponding to the spectral decomposition of L_ε^*, where $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and the Fourier coefficients a_k are obtained from the boundary condition $\mu(x,0) = \mu(-x,\frac{1}{2}), x \in \mathbb{R}$.

Here is the key observation opening the route towards finding local maxima of the SPA coefficient. The terms in the sum (8) decay in time exponentially fast with rates λ_k, and therefore the terms corresponding to larger eigenvalues contribute less than the ones belonging to the low lying eigenvalues. This underlines their key importance. Fortunately, in the case of a double well potential the following theorem holds.

Theorem 1 ('spectral gap'). *In the limit of small noise, the following asymptotic properties are valid:*

$$\begin{aligned}
&\lambda_0 = \lambda_0(\varepsilon) = 0, \ \textit{and}\ \Psi_0 = e^{-\frac{2U_1}{\varepsilon}},\\
&\lambda_1 = \lambda_1(\varepsilon) = \frac{1}{2\pi}\sqrt{U_1''(1)|U_1''(0)|}\cdot e^{-v/\varepsilon}(1+\mathcal{O}(\varepsilon)),\\
&\lambda_2 = \lambda_2(\varepsilon) \geq C > 0 \ \textit{uniformly in}\ \varepsilon.
\end{aligned}$$

Moreover, we can provide a very good approximation to the first eigenfunction Ψ_1. Let $-1 < x_- < 0 < x_+ < 1$ such that $U_1(x_-) = -\frac{V}{3}$ and $U_1(x_+) = -\frac{v}{3}$. Fix also some v' such that $\frac{2}{3} < \frac{v'}{v} < 1$. Then the following holds.

Theorem 2. *In the limit of small noise, the first eigenfinction of L_ε^* is found as $\Psi_1 = \Phi_1 e^{-\frac{2U_1}{\varepsilon}}$, where*

$$\max_{x \le x_-} |\Phi_1 - a(\varepsilon)| \le e^{-\frac{2V}{3\varepsilon}},$$

$$\max_{x_- \le x \le x_+} |\Phi_1 - a(\varepsilon) - (1 - a(\varepsilon)) \frac{\int_{-1}^{x} e^{\frac{2U(y)}{\varepsilon}} dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}} dy}| \le e^{-\frac{v'}{\varepsilon}},$$

$$\max_{x \ge x_+} |\Phi_1 - 1| \le e^{-\frac{2v}{3\varepsilon}}$$

with $a(\varepsilon) = -\sqrt{\frac{U_1''(-1)}{U_1''(1)}} e^{-\frac{V-v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon))$.

The result of Theorem 1 plays a crucial role in our analysis. There is a *spectral gap* between the first eigenvalue and the rest of the spectrum. Consequently, only the first two terms of (8) can have an essential contribution to the SPA coefficient η^X.

2.2 Asymptotics of the SPA coefficient

The following theorem gives the asymptotics of the first two Fourier coefficients a_0 and a_1 in the Floquet type expansion of the previous subsection.

Theorem 3. *We have*

$$a_0 = \|\Psi_0\|,$$
$$a_1 = \frac{\|\Psi_1\|}{\|\Psi_0\|^2} \cdot \frac{\langle \Psi_0(-\cdot), \Psi_1 \rangle}{\|\Psi_1\|^2 - e^{-\frac{1}{2}T\lambda_1} \langle \Psi_1(-\cdot), \Psi_1 \rangle} + r,$$

where r vanishes in the limit of small noise and for $T \ge \exp\{(v+\delta)/\varepsilon\}$, δ being positive and sufficiently small.

Recall the definition (4) of the SPA coefficient. Denote

$$S^X(\varepsilon, T) = \int_0^{\frac{1}{2}} \mathbf{E}_\mu X_{Ts}^{\varepsilon,T} \cdot e^{2\pi i s} \, ds.$$

In these terms we identify $\eta^X = 4|S^X|^2$.

Theorem 4. *Let $T \ge \exp\{(v+\delta)/\varepsilon\}$ for δ positive and sufficiently small. Then the following expansion for S^X holds in the small noise limit $\varepsilon \to 0$*

$$S^X = \frac{1}{\pi i} b_0 + \frac{1}{\pi i - \frac{1}{2}\lambda_1 T} b_1 + r_1$$

where the rest term r_1 tends to zero with ε and the coefficients are given by

$$b_0 = \frac{\int y \, e^{-\frac{2U_1(y)}{\varepsilon}} dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}} dy},$$

$$b_1 = -\frac{1 + e^{-\frac{1}{2}T\lambda_1}}{2} \cdot \frac{\int y \, \Psi_1(y) \, dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}} dy} \cdot \frac{\langle \Psi_0(-\cdot), \Psi_1 \rangle}{\|\Psi_1\|^2 - e^{-\frac{1}{2}T\lambda_1} \langle \Psi_1(-\cdot), \Psi_1 \rangle}.$$

Finally,

$$\eta^X = b_0^2 \frac{4}{\pi^2} \frac{(\lambda_1 T)^2}{4\pi^2 + (\lambda_1 T)^2} + R, \tag{9}$$

where R tends to zero with ε.

Let us now study the *resonance* behaviour of the SPA coefficient η^X, i.e. investigate whether it has a local maximum in ε. We formulate the following Lemma which is obtained by application of Laplace's method of asymptotic expansion of singular integrals, see [Erd56, Olv74] or also [Pav02, IP02].

Lemma 1 ('Laplace's method'). *In the small noise limit, the following holds true:*

$$\begin{aligned} b_0 &= -1 - \frac{1}{4} \frac{U_1^{(3)}(-1)}{U_1''(-1)^2} \varepsilon + \mathcal{O}(\varepsilon^2), \\ b_1 &= -1 + \mathcal{O}(\varepsilon), \end{aligned}$$

and consequently

$$\begin{aligned} b_0^2 &= 1 + \frac{1}{2} \frac{U_1^{(3)}(-1)}{U_1''(-1)^2} \varepsilon + \mathcal{O}(\varepsilon^2), \\ (b_0 - b_1)^2 &= \mathcal{O}(\varepsilon^2). \end{aligned} \tag{10}$$

The following Theorem exhibits the defect of the notion of spectral power amplification for our diffusions in periodically and discontinuously switching potential states.

Theorem 5. *Let us fix δ positive and sufficiently small and $\Delta > v + \delta$. Let also $U_1(x) - 2U_1(-x) < v + V$ for all $x \in \mathbb{R}$ (no strong asymmetry!). Then for $T \to \infty$ and ε satisfying*

$$\frac{v + \delta}{\ln T} \le \varepsilon \le \frac{\Delta}{\ln T}$$

the following asymptotic expansion for the SPA coefficient holds:

$$\eta^X(\varepsilon, T) = \frac{4}{\pi^2} \left(1 + \frac{1}{2} \frac{U_1^{(3)}(-1)}{U_1''(-1)^2} \varepsilon \right) + \mathcal{O}\left(\frac{1}{\ln^2 T} \right).$$

This result has the following surprising consequences.

Corollary 1. *For $T \to \infty$ and $\varepsilon \in [\frac{v+\delta}{\ln T}, \frac{\Delta}{\ln T}]$ the SPA coefficient is a decreasing function of ε if $U_1^{(3)}(-1) < 0$ and an increasing function of ε if $U_1^{(3)}(-1) > 0$.*

Thus, the SPA coefficient as quality measure for tuning shows *no resonance* in a domain above Freidlin's threshold for quasi-deterministic periodicity (see [Fre00]). This contradicts the physical intuition for the 'effective dynamics'. The reason for this surprising phenomenon can only be hidden in the *intra-well* behaviour of the diffusion neglected when passing to the reduced Markov chain. We return to this question later. Let us next study in more detail the 'effective dynamics' of the diffusion (6).

2.3 The 'effective dynamics': two-state Markov chain

The idea of approximation of diffusions in potential landscapes by appropriate finite state Markov chains in the context of stochastic resonance was suggested by Eckmann and Thomas [ET82], and C. Nicolis [Nic82], and developed by McNamara and Wiesenfeld [MW89]. In this section we follow [Pav02, IP02]. The discrete time case was studied in [IP01].

In order to catch the main features of the spatial *inter-well* transitions of the diffusion (6) we consider the time inhomogeneous Markov chain $Y^{\varepsilon,T}$ living on the diffusion's meta-stable states ± 1. The infinitesimal generator of $Y^{\varepsilon,T}$ is periodic in time and is given by

$$Q_{\varepsilon,T}(t) = \begin{cases} \begin{pmatrix} -\varphi & \varphi \\ \psi & -\psi \end{pmatrix}, & \frac{t}{T} \pmod 1 \in [0, \frac{1}{2}), \\ \begin{pmatrix} -\psi & \psi \\ \varphi & -\varphi \end{pmatrix}, & \frac{t}{T} \pmod 1 \in [\frac{1}{2}, 1). \end{cases} \tag{11}$$

The transition rates φ and ψ which are responsible for the similarity of the two processes are chosen to be exponentially small in ε:

$$\varphi = \frac{1}{2\pi}\sqrt{U_1''(-1)|U_1''(0)|} \cdot e^{-V/\varepsilon} \text{ and } \psi = \frac{1}{2\pi}\sqrt{U_1''(1)|U_1''(0)|} \cdot e^{-v/\varepsilon}.$$

To exponential order they correspond (as they should) to the inverses of the Kramers-Eyring transition times. The invariant measure of $Y^{\varepsilon,T}_{Tt}$ can be obtained as a solution of a forward Kolmogorov equation and is given by

$$\begin{aligned} \nu^-(t) &= \frac{\psi}{\varphi+\psi} + \frac{\varphi-\psi}{\varphi+\psi}\frac{e^{-(\varphi+\psi)Tt}}{1+e^{-\frac{1}{2}(\varphi+\psi)Tt}}, \\ \nu^+(t) &= \frac{\varphi}{\varphi+\psi} - \frac{\varphi-\psi}{\varphi+\psi}\frac{e^{-(\varphi+\psi)Tt}}{1+e^{-\frac{1}{2}(\varphi+\psi)Tt}}, \quad t \in [0, \tfrac{1}{2}], \end{aligned} \tag{12}$$

and $\nu^\pm(t) = \nu^\mp(t+\frac{1}{2})$ for $t \geq 0$.

We define the SPA coefficient η^Y for the Markov chain $Y^{\varepsilon,T}$ analogously to (4). In this much simpler setting given it can be described explicitly.

Theorem 6. *For all $\varepsilon > 0$ and $T > 0$ the following holds:*

$$\eta^Y(\varepsilon, T) = \frac{4}{\pi^2}\frac{T^2(\varphi-\psi)^2}{4\pi^2 + T^2(\varphi+\psi)^2}. \tag{13}$$

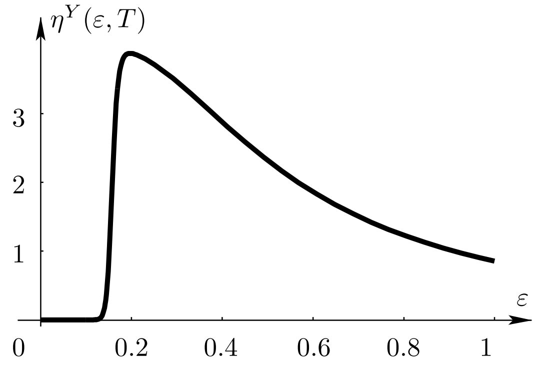

Fig. 4. SPA coefficient η^Y of the two-state Markov chain.

Compare (13) with (9). Since $(\varphi \pm \psi)^2 \approx \lambda_1^2$ in the limit of small ε, the formulae for η^X and η^Y differ only in the 'geometric' pre-factor b_0^2 and the asymptotically negligible remainder term R.

The explicit formula (13) allows to study the local maxima of η^Y as a function of noise intensity for large periods T in great detail (see Fig. 4).

Theorem 7. *In the limit $T \to \infty$ the function $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ has a local maximum at*

$$\varepsilon(T) \approx \frac{v+V}{2} \frac{1}{\ln T},$$

or in the limit $\varepsilon \to 0$ in terms of T

$$T(\varepsilon) \approx \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V-v}} e^{\frac{V+v}{\varepsilon}}.$$

The 'resonance' behaviours of η^X and η^Y are quite different. Whereas the diffusion's SPA has no extremum for small ε, the Markov chain's *always* has. What can be responsible for this discrepancy? Note that the Markov chain mimics only the *inter-well* dynamics of the diffusion. Thus, the SPA coefficient η^Y measures only the spectral energy contributed by inter-well jumps. On the other hand, η^X also counts the numerous *intra-well* fluctuations of the diffusion the weight of which evidently becomes overwhelming in the small noise limit. These fluctuations have small energy. But since the diffusion spends most of its time near ± 1 the local asymmetries of the potential at these points dominate the picture and destroy optimal tuning.

To underpin this heuristics mathematically, let us now make the idea of neglecting the diffusion's intra-well fluctuations precise. For example, we *cut off* those among them which have not enough energy to reach half the height of the potential barrier between the wells. Consider the cut-off function g defined by

$$g(x) = \begin{cases} -1, & x \in [x_1, x_2], \\ 1, & x \in [y_1, y_2], \\ x, & \text{otherwise}, \end{cases}$$

where $x_1 < -1 < x_2 < 0$ and $0 < y_1 < 1 < y_2$ are such that $U_1(x_1) = U_1(x_2) = -\frac{V}{4}$ and $U_1(y_1) = U_1(y_2) = -\frac{v}{4}$, see Fig. 5. Now we study the

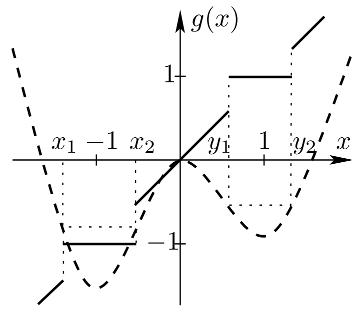

Fig. 5. Function g designed to cut off diffusion's intrawell dynamics.

modified SPA coefficient of a diffusion defined by

$$\widetilde{\eta}^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu \left[g(X^{\varepsilon,T}_{Ts}) \right] e^{2\pi i s} \, ds \right|^2 .$$

Following the steps of Subsection 2.2 we obtain a formula for $\widetilde{\eta}^X$ which is quite similar to (9) and (13):

$$\widetilde{\eta}^X(\varepsilon, T) = \widetilde{b}_0^2 \frac{4}{\pi^2} \frac{(\lambda_1 T)^2}{4\pi^2 + (\lambda_1 T)^2} + \widetilde{R},$$

where $\widetilde{R}$ is a small remainder term, and

$$\widetilde{b}_0^2 = \left(\frac{\int g(y) e^{-\frac{2U_1(y)}{\varepsilon}} \, dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}} \, dy} \right)^2 = 1 - 4 \sqrt{\frac{U_1''(-1)}{U_1''(1)}} e^{-\frac{V-v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon))$$

(compare to (10)).

The modified geometric pre-factor $\widetilde{b}_0^2$ is essentially smaller than its counterpart b_0^2. This has crucial influence on the SPA coefficient $\widetilde{\eta}^X$: in the limit of large period and small noise its behaviour now reminds of η^Y.

Theorem 8. *Let the assumptions of Theorem 5 hold. Then for any $\gamma > 1$ in the limit $T \to \infty$ the function $\varepsilon \mapsto \widetilde{\eta}^X(\varepsilon, T)$ has a local maximum on*

$$\left[\frac{1}{\gamma} \frac{v + V}{2} \frac{1}{\ln T}, \gamma \frac{v + V}{2} \frac{1}{\ln T} \right].$$

In other words, the optimal tuning for the measure of goodness $\widetilde{\eta}^X$ *exists and is given approximately by*

$$\varepsilon(T) \approx \frac{v+V}{2}\frac{1}{\ln T}.$$

3 Smooth periodic potentials and a robust resonance notion

The serious defect of the SPA coefficient in the prediction of the stochastic resonance point in the Markov chain models containing the effective dynamics of complex diffusion models motivates us to look for *robust* notions of quality of periodic tuning. Since the dynamics of the Markov chain only retains the rough mechanism of transitions between the domains of attraction given in the underlying potential landscape, such a notion should only take into account the most important aspects of the *attractor hopping.* Also, as the alternative notions discussed in the preceding section show, the resonance point is by no means a canonical object, independent of the way tuning quality is measured. We think that the methods of advanced large deviations' theory behind the notion to be explained in this section will give it a more natural place, and possibly qualify it as canonical.

At the same time, we essentially generalize the simplified model of time periodic potential considered in the previous section, and lift the study of stochastic resonance to a somewhat more abstract level. The potential function U in the present section will still be supposed to be one-dimensional in space. But its periodic time variation will just be assumed to be continuous, and otherwise quite general. More precisely, we study diffusion processes driven by a Brownian motion of intensity ε given by the stochastic differential equation

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{T})\,dt + \sqrt{\varepsilon}\,dW_t, \quad t \geq 0.$$

The underlying potential landscape (see Fig. 6) is described by a function $U(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, which is periodic in time with period 1, and its temporal variation, by the rescaling with very large T, acts on the diffusion at a very small frequency. U is supposed to have exactly two wells located at ± 1, separated by a saddle at 0. The depth of $U(\cdot,t)$ at ± 1 is given by the 1-periodic depth functions $\frac{1}{2}D_{\pm 1}(t)$ which are assumed to never fall below zero. Let us now look at exponential time scales ρ, related to the natural time scale T by $T = e^{\rho/\varepsilon}$. In this setting, Freidlin's theory of quasi-deterministic motion indicates that transitions e.g. from the domain of attraction of -1 to the domain of attraction of 1 will occur as soon as D_{-1} gets less than ρ, i.e. at time

$$a_\rho^{\pm 1} = \inf\{t \geq 0 : D_{\pm 1}(t) \leq \rho\}.$$

This triggers periodic behaviour of the diffusion trajectories on long time scales. The modern theory of meta-stability in *time homogeneous diffusion*

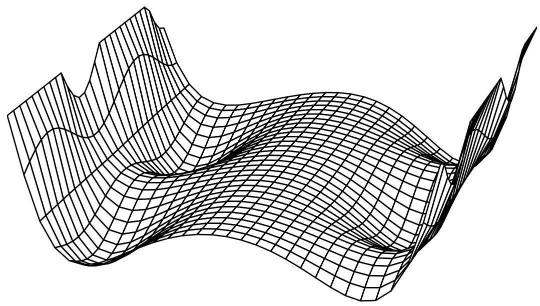

Fig. 6. Potential landscape U.

processes produces the exponential decay rates of transition probabilities between different domains of attraction of a potential landscape together with very sharp multiplicative error estimates, uniformly on compacts in system parameters. Their sharpest forms are presented in some very recent papers by Bovier et al. [BEGK02, BGK02]. We use this powerful machinery in order to obtain very precise estimates of the exponential tails of the laws of the transition times between domains of attraction. To this end, we have to extend the estimates by Bovier et al. [BGK02] to the framework of *time inhomogeneous diffusions*. In the underlying one-dimensional situation, this can be realized by freezing the time dependence of the potential on small time intervals to define lower and upper bound time homogeneous potentials not differing very much from the original one. Comparison theorems are used to control the transition behaviour from above and below through the corresponding time homogeneous diffusions. This allows very precise estimates on the probabilities with which the diffusion at time scale $T = e^{\rho/\varepsilon}$ transits from the domain of attraction of -1 to the domain of attraction of 1 or vice versa within time windows $[(a_\rho^{-1} - h)T, (a_\rho^{-1} + h)T]$ for small $h > 0$. If $\tau_x(X^{\varepsilon,T})$ denotes the transit time to x, it is given by

$$\lim_{\varepsilon \to 0} \varepsilon \ln (1 - M(\varepsilon, \rho)) = \max_{i=\pm 1} \left\{ \rho - D_i(a_\rho^i - h) \right\},$$

with

$$M(\varepsilon, \rho) = \min_{i=\pm 1} P_i(\tau_{-i}(X^{\varepsilon,T}) \in [(a_\rho^i - h)T, (a_\rho^i + h)T]), \quad \varepsilon > 0, \rho \in I_R,$$

and where I_R is the *resonance interval*, i.e. the set of scale parameters for which trivial or chaotic transition behaviour of the trajectories is excluded. The stated convergence is *uniform* in ρ on compact subsets of I_R. This allows us to take $M(\varepsilon, \rho)$ as our measure of periodic tuning, compute the scale $\rho_0(h)$ for which the transition rate is optimal, and define the *stochastic resonance point* as the eventually existing limit of $\rho_0(h)$ as $h \to 0$. This resonance notion has the big advantage of being robust for the passage from the diffusion to the two state Markov chain describing the effective dynamics.

3.1 Transition times for the Markov chain

Let us first discuss the effective dynamics modelled by a continuous time two state Markov chain. The states represent the positions of the bottoms of the wells of the double well potential. The transition rates picture the transition mechanism of the diffusion to which we return later. We shall first define the interval of time scales for which transitions are not trivial.

Definition of the resonance interval

Let us consider the time continuous Markov chain $Y^{\varepsilon,T} = (Y_t^{\varepsilon,T})_{t\geq 0}$ taking values in the state space $\{-1,1\}$ with initial data $Y_0^{\varepsilon,T} = -1$. Suppose that the infinitesimal generator is given by

$$G_{\varepsilon,T}(t) = \begin{pmatrix} -\varphi(\frac{t}{T}) & \varphi(\frac{t}{T}) \\ \psi(\frac{t}{T}) & -\psi(\frac{t}{T}) \end{pmatrix},$$

where T is an exponentially large time scale (we recall that $T = e^{\rho/\varepsilon}$, $\rho > 0$), ψ and φ are 1-periodic functions describing a rate which just produces the transition dynamics of the diffusion between the potential minima ± 1. Let us recall that, if we consider some time-independent potential U, then the mean transition time between the wells is given by the Kramers-Eyring law. If the diffusion starts in the minimum of one well, the mean exit time is equivalent to $e^{V/\varepsilon}$, where $\frac{V}{2}$ is the height of the barrier separating the two minima of the potential. Consequently the transition rate should be proportional to $e^{-V/\varepsilon}$.

In the framework we now consider the depth of the wells depends continuously on time. In this situation it is natural to postulate the following periodic infinitesimal probabilities

$$\varphi(t) = e^{-D_{-1}(t)/\varepsilon}, \quad t \geq 0. \tag{14}$$

Let us assume that $D_1(t) = D_{-1}(t+\alpha)$, $t \geq 0$, with phase shift $\alpha \in (0,1)$ and

- all local extrema of $D_{\pm 1}(\cdot)$ are global;
- the functions $D_{\pm 1}(\cdot)$ are strictly monotonous between the extrema.

Hence $\psi(t) = \varphi(t+\alpha)$, $t \geq 0$, and

$$\psi(t) = e^{-D_1(t)/\varepsilon}, \quad t \geq 0. \tag{15}$$

Let us define S_{-1} to be the normalized time of the first jump from the state -1 to 1, i.e. $S_{-1} = \inf\{t \geq 0 : Y_{tT}^{\varepsilon,T} = 1\}$ Analogously, S_1 will be the time of first jump from state 1 to -1, starting with $Y_0 = 1$. We are especially interested in the behaviour of S as T becomes very large, that is as $\varepsilon \to 0$. In fact we get the following dichotomy of possible behaviour:

- If $\rho > \inf_{t\geq 0} D_{-1}(t)$, the law of S_{-1} tends to the Dirac measure in the point a_ρ^{-1} given by

$$a_\rho^{\pm 1} = \inf\{t \geq 0 : D_{\pm 1}(t) \leq \rho\}. \tag{16}$$

- If $\rho \leq \inf_{t\geq 0} D_{-1}(t)$, then the probability measure of S_{-1} tends weakly to the null measure.

It suffices to replace D_{-1} by D_1 and a_ρ^{-1} by a_ρ^1 to obtain similar results for S_1.

This leads to the following interpretation:

If $\rho \geq D_{-1}(0)$, that is, if the time scale T is very large, then on this exponential scale, the asymptotic behaviour of the Markov chain is characterized by an instantaneous jump, i.e. $a_\rho^{-1} = 0$. This just means that a clock ticking in units of T will record a jump of the process as instantaneous, since it occurs on a smaller scale.

In case $\rho < \inf_{t\geq 0} D_{-1}(t)$, the time scale T is too small compared to the transition rates. Consequently no transitions will be observed, and the process never jumps on this scale.

In the last case $D_{-1}(0) > \rho > \inf_{t\geq 0} D_{-1}(t)$. So the infinitesimal probability at time 0 is too small to allow any transition, and the Markov chain will have to wait until this probability is large enough to allow for jumps, that is approximatively $a_\rho T$. This case is the only interesting case, in the sense that the chain stays for some time in the starting state before it jumps to the other one.

To observe stochastic resonance we obviously need to study both transitions from -1 to 1 and vice versa. So we define some interval I_R called *interval of resonance* (see Fig. 7) which is to contain those exponential scales in which the process on the one hand asymptotically cannot always stay in the same state with positive probability, and on the other hand asymptotically cannot jump instantaneously from one state to the other.

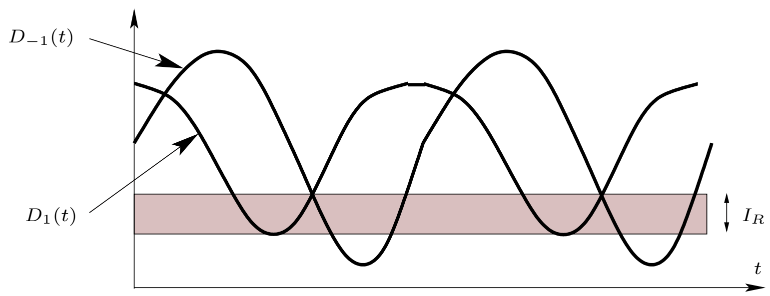

Fig. 7. Resonance interval I_R:

$$I_R = (\max_{i=\pm 1} \inf_{t\geq 0} D_i(t), \inf_{t\geq 0} \max_{i=\pm 1} D_i(t)).$$

Optimal tuning for the Markov chain

Let us now assume that we are in the range of non-trivial jumping, that is $\rho \in I_R$. We next determine an optimal tuning rate or stochastic resonance point. It will be based on the density of the first jump, in particular the intensity of its peak, which we propose as a new measure of quality of tuning. For $h > 0$ we shall compare the probabilities with which the first transition takes place within the window of exponential length $[(a_\rho^i - h)T, (a_\rho^i + h)T]$, $i = \pm 1$, for different ρ, maximize this quantity in ρ and finally take the window length to 0. More formally, for $h > 0$ small enough define

$$N(\varepsilon, \rho) = \min_{i=\pm 1} \mathbf{P}_i(S_i \in [(a_\rho^i - h)T, (a_\rho^i + h)T]),\ \varepsilon > 0,\ \rho \in I_R, \tag{17}$$

and call it *transition probability for a time window of width h* for the Markov chain. The optimal parameter ρ_0 will tell us at which time scale it is most likely to see trajectories of the chain with first jump in the corresponding window, and further jumps in accordingly displaced windows. In particular, it will tell us at which scale periodic trajectories of just this period are most probable. Since the probability density of the first transition times from one state to the other is well known, for example the density of S_{-1} equals

$$p(t) = \varphi(t) e^{-\int_0^t \varphi(s) ds},$$

we can compute an explicit expression for $N(\varepsilon, \rho)$. The optimal time scale will be determined by a combination of a large deviations result concerning the first jump of the Markov chain parametrized by the logarithmic scale ρ of time, and a maximization problem for the large deviation rates in ρ to which the transition probabilities converge uniformly.

Using Laplace's method to estimate the singular integrals appearing as $\varepsilon \to 0$, we obtain the required asymptotic result.

Theorem 9. *Let Γ be a compact subset of I_R, $h_0 < \max\{a_\rho^{-1}, \frac{T}{2} - a_\rho^{-1}\}$. Then for $0 < h \leq h_0$*

$$\lim_{\varepsilon\to 0} \varepsilon \ln(1 - N(\varepsilon, \rho)) = \max_{i=\pm 1} \left\{\rho - D_i(a_\rho^i - h)\right\} \tag{18}$$

uniformly for $\rho \in \Gamma$.

Since the convergence is uniform in ρ, it suffices to minimize the left hand side of (18) to obtain an optimal tuning point. For h small the eventually existing global minimizer $\rho_R(h)$ of

$$I_R \ni \rho \mapsto \max_{i=\pm 1} \left\{\rho - D_i(a_\lambda^i - h)\right\}$$

is a good candidate for our resonance point. But it still depends on h. To get rid of this dependence, we shall consider the limit of $\lambda_R(h)$ as $h \to 0$.

Definition 1. *Suppose that*

$$I_R \ni \rho \mapsto \max_{i=\pm 1} \left\{ \rho - D_i(a^i_\rho - h) \right\}$$

possesses a global minimum $\rho_R(h)$. Suppose further that

$$\rho_R = \lim_{h \to 0} \rho_R(h)$$

exists in I_R. We call ρ_R the stochastic resonance point *of the Markov chain $Y^{\varepsilon,T}$ with time periodic infinitesimal generator $G_{\varepsilon,T}$.*

In fact the stochastic resonance point exists if one of the depth functions, and therefore both, due to the phase lag, has a unique point of maximal decrease in the interval in which it is strictly decreasing.

Example: In fact all the results presented before, in the case of a time dependent potential U with meta-stable states at ± 1 also hold true if the meta-stable states are allowed to move periodically but stay away from the saddle 0. Then the state -1 of the Markov chain represents the left meta-stable state and 1 represents the right one. We shall mention one classical example in stochastic resonance (see, for instance [GHJM98]) which is the over-damped motion of a Brownian particle in the potential

$$2U(x,t) = V(x) + Ax\cos(2\pi t),$$

where V denotes a reflection-symmetric potential with two wells located at ± 1. In this particular case, for $0 < A < V(0) - V(-1)$,

$$D_{\pm 1}(t) = V(0) - V(-1) \pm A\cos(2\pi t).$$

Hence the phase lag α is equal to π and the resonance interval is

$$I_R = (V(0) - V(-1) - A, V(0) - V(-1)).$$

Let $h > 0$ small enough, then the logarithmic time scale which asymptotically optimizes the quality measure $N(\varepsilon, \rho)$ is given by

$$\rho_R(h) = V(0) - V(-1) - A\sin(\pi h).$$

In order to obtain the resonance point, we just let h tend to zero, to obtain $\rho_R = \lim_{h\to 0} \rho_R(h) = V(0) - V(-1)$, that is the average depth of the time periodic potential U. In this particular case, it is obvious that the resonance point coincides with the point of maximal decrease of the depth functions D_{-1} and D_1. This example is treated in detail in [HI02].

3.2 Transition times for the diffusion and robustness

As seen in the preceding subsection, for the effective dynamics we obtain both simple and explicit results. Now we shall show how our measure of quality based purely on the jumps for the two-state Markov chain can be extended to the diffusion case. We just have to generalize the notion of jumps to the transition times between the two domains of attraction of the potential landscape, i.e. the two wells. The accordingly generalized measure of quality of periodic tuning possesses the desired property of being robust. The analogous notion of interval of resonance will then be presented in the following subsection. In our presentation we follow [HI].

Resonance interval for diffusions

Recall that the underlying potential is described by a function $U(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, such that $U'(\cdot,\cdot)$ is both continuous in time and space. The local minima are located at ± 1 and the saddle point at 0, independently of time. Our main concern will be the asymptotics of the transition times from the domain of attraction $(-\infty, 0)$ of -1 to the domain of attraction $(0,\infty)$ associated with 1 of the time inhomogeneous diffusion in the small noise limit $\varepsilon \to 0$. More precisely we will be interested in describing the exponential transition rate from the domain of attraction of -1 to the domain of attraction of 1. Our potential not being time homogeneous, we shall make use of comparison arguments with diffusions possessing time independent potentials in order to perform a careful reduction of the inhomogeneous exit problem to the homogeneous one, and use the asymptotic results well known for this particular case. This will be achieved by freezing the driving force derived from the potential on small time intervals on the minimal or maximal level it takes there. To be more precise, for each interval $I \subset \mathbb{R}_+$ let

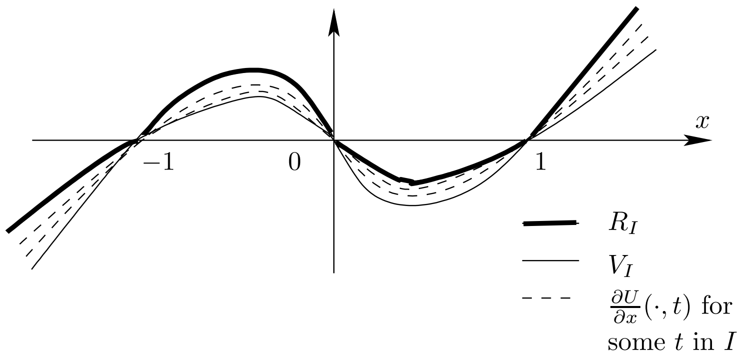

Fig. 8. Definition of V_I and R_I.

$$V_I(x) = \sup_{t\in I} \frac{\partial U}{\partial x}(t,x) \quad \text{and} \quad R_I(x) = \inf_{t\in I} \frac{\partial U}{\partial x}(t,x). \tag{19}$$

The regularity conditions valid for U imply that V and R are continuous functions. Moreover $V_I(-1) = R_I(-1) = 0$, see Fig. 8. If $I = [a, b]$, we denote by $\overline{X}^{\varepsilon,I}$ the solution of the SDE on $\mathbb{R}_+$

$$\begin{cases} d\overline{X}_t^{\varepsilon,I} = -R_I(\overline{X}_t^{\varepsilon,I})\,dt + \sqrt{\varepsilon}\,dW_t, \\ \overline{X}_0^{\varepsilon,I} = X_{aT}^{\varepsilon,T}. \end{cases} \tag{20}$$

$\underline{X}^{\varepsilon,I}$ is defined in the same way replacing R_I by V_I. These two *time homogeneous* diffusions are used to control the *time inhomogeneous* diffusion $X^{\varepsilon,T}$ as long as time runs in the interval I. In fact, we have P-a.s.

$$\underline{X}_{tT}^{\varepsilon,I} \le X_{(t+a)T}^{\varepsilon,T} \le \overline{X}_{tT}^{\varepsilon,I}, \quad t \in [0, b-a].$$

Hence in order to study the time the diffusion needs to reach 1 starting in the left well, we shall consider the diffusion on one period. This time interval can be decomposed into finitely many small time intervals I_n, $0 \le n \le n_0$. We shall then freeze the potential on I_n and analyze if the the diffusions $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$ have enough time in I_n to reach the top of the barrier between the two wells and, consequently on the same scale reach 1, the bottom of the right well. In other words we need to get information on the exit problem for the homogeneous diffusions $\underline{X}^{\varepsilon,I}$ and $\overline{X}^{\varepsilon,I}$.

We shall refer to the most recent and advanced development of sharp estimates for transition times presented in Bovier et al. [BEGK02] and [BGK02]. They are valid far beyond our modest framework, and we just present the results we will use here. For this purpose, suppose that $U_1(\cdot)$ is a purely space dependent $\mathcal{C}^2$ potential function of the shape similar to those on Fig. 3. It possesses only ± 1 as local minima, separated by the saddle 0. Suppose that the curvature of U_1 at -1 is strictly positive, i.e. $U_1''(-1) > 0$. As for ultra- or hypercontractivity type properties for U_1, we shall assume that it has exponentially tight level sets, i.e. there is $M_0 > 0$ such that for any $M \ge M_0$ there exists a constant $C(M)$ such that for $\varepsilon \le 1$

$$\int_{\{y: U_1(z) \ge M\}} e^{-2U_1(z)/\varepsilon}\,dz < C(M) e^{-M/\varepsilon}. \tag{21}$$

We shall concentrate in this situation on an exit of the domain of attraction of the meta-stable point -1 for the diffusion associated with the SDE

$$\begin{cases} dX_t^\varepsilon = -U_1'(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t, \\ X_0^\varepsilon = x < 0. \end{cases}$$

We are interested in the asymptotics of the first time X^ε reaches 1:

$$\tau_1(X^\varepsilon) = \inf\{t > 0 : \ X_t^\varepsilon = 1\}.$$

Then we obtain the following result.

Theorem 10. *Let $\lambda(\varepsilon)$ denote the principal eigenvalue of the linear operator*

$$L_\varepsilon u = \frac{\varepsilon}{2}u'' - U_1'u', \quad u \in \mathcal{L}^2((-\infty, 1], e^{-2U_1/\varepsilon}dx)$$

with Dirichlet boundary conditions at 1. *Then for every compact $K \subseteq (-\infty, 0)$ there is a constant $c > 0$ such that*

$$\mathbf{P}_x(\tau_1(X^\varepsilon) > t) = e^{-\lambda(\varepsilon)t}(1 + \mathcal{O}_K(e^{-c/\varepsilon})), \tag{22}$$

where $\mathcal{O}_K$ denotes an error term which is uniform in $x \in K$, $t \geq 0$. Moreover, for the asymptotic behaviour of the eigenvalue $\lambda(\varepsilon)$ the following holds

$$\lambda(\varepsilon)\mathbf{E}_x[\tau_1(X^\varepsilon)] \to 1 \quad \text{uniformly on compacts } K \subseteq (-\infty, 0) \text{ as } \varepsilon \to 0. \tag{23}$$

Large deviations' theory reveals the asymptotic behaviour of the principal eigenvalue:

$$\lim_{\varepsilon\to 0} \varepsilon \ln \lambda(\varepsilon) = -2(U_1(0) - U_1(-1)).$$

This allows us to deduce that the mean hitting time $\mathbf{E}_x[\tau_1(X^\varepsilon)]$ is equivalent to $e^{\frac{2}{\varepsilon}(U_1(0)-U_1(-1))}$ as $\varepsilon \to 0$. Here $U_1(0)-U_1(-1)$ is the depth of the starting well. Moreover, by Theorem 10, the normalized hitting time $\frac{\tau_1(X^\varepsilon)}{\mathbf{E}_x[\tau_1(X^\varepsilon)]}$ converges in law to an exponential random variable with mean 1 as $\varepsilon \to 0$.

These results are very precise. They describe the asymptotic time of the barrier crossing and at the same time give an estimation of the probability to cross the barrier in a small time window around this asymptotic deterministic time. We can apply them to the 'frozen' potential $U(\cdot, \cdot)$ on the small time intervals I_n. We thereby assume for simplicity that the frozen potentials are regular of order $\mathcal{C}^2$. Let us choose $n \geq 0$ and set $I_n = [r_n, r_{n+1}]$. We assume that $X^{\varepsilon,T}$ has not reached the top of the barrier before r_nT and study what happens during the time interval $[r_nT, r_{n+1}T]$. We have already seen that $X^{\varepsilon,T}$ is controlled by both $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$. On the one hand, it suffices to prove that $\underline{X}^{\varepsilon,I_n}$ reaches 1 before $r_{n+1}T$ in order to get $\tau_1(X^{\varepsilon,T}) \leq r_{n+1}T$. On the other hand, if we get that $\overline{X}^{\varepsilon,I_n}$ does not hit 1 then so does $X^{\varepsilon,T}$. As $\varepsilon \to 0$, Theorem 10 tells us, for example, that the probability that $\underline{X}^{\varepsilon,I_n}$ reaches 1 before $r_{n+1}T$ is close to 1 if the depth of the left well is smaller than $\lim_{\varepsilon\to 0} \varepsilon \ln(r_{n+1} - r_n)T = \rho$. Indeed we get $\lim_{\varepsilon\to 0}(r_{n+1} - r_n)\lambda(\varepsilon)T = +\infty$ which implies by (22) that

$$\lim_{\varepsilon\to 0} \mathbf{P}_x(\tau_1(\underline{X}^{\varepsilon,I_n}) > (r_{n+1} - r_n)T) = 0.$$

The statements depend weakly on the depth of the well of the potential associated with $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$. Since $\frac{\partial U}{\partial x}$ is continuous both in x and t, if we choose the length of all intervals I_n small enough then the well depth functions associated with the two time homogeneous diffusions are equivalent to $D_{-1}(r_n)$, the depth of the left well of the landscape U. Hence the diffusion

$X_{tT}^{\varepsilon,T}$ reaches 1 asymptotically as soon as the depth $D_{-1}(t)$ goes below the level ρ. This means

$$\lim_{\varepsilon\to 0}\frac{\tau_1(X^{\varepsilon,T})}{T}=a_\rho^{-1},$$

where a_ρ^{-1} was defined in (16).

Knowing the asymptotics of the time at which the diffusion reaches the barrier separating the two wells in order to hit 1 puts us again in a position in which we can discuss a resonance interval as for the reduced model. We obtain the same interval

$$I_R=(\max_{i=\pm 1}\inf_{t\geq 0}D_i(t),\inf_{t\geq 0}\max_{i=\pm 1}D_i(t)).$$

Optimal tuning for the diffusion and robustness

The comparison between time inhomogeneous and homogeneous potentials and the asymptotic result 10 enable us to proceed to the completion of our approach of stochastic resonance for diffusions. We have very precise estimates on the probabilities with which the diffusion at time scale $T=e^{\rho/\varepsilon}$ transits from the domain of attraction of -1 to the domain of attraction of 1 and vice versa within the time windows $[(a_\rho^i-h)T,(a_\rho^i+h)T]$ for small $h>0$. On their basis we may define a measure of quality of tuning for the diffusion which corresponds to (17):

$$M(\varepsilon,\rho)=\min_{i=\pm 1}\mathbf{P}_i(\tau_{-i}(X^{\varepsilon,T})\in[(a_\rho^i-h)T,(a_\rho^i+h)T]),\quad \varepsilon>0,\rho\in I_R. \quad (24)$$

We may now state our main result on uniform transition rates.

Theorem 11. *Let Γ be a compact subset of I_R, $h_0>0$ small enough. Then*

$$\lim_{\varepsilon\to 0}\varepsilon\ln(1-M(\varepsilon,\rho))=\max_{i=\pm 1}\left\{\rho-D_i(a_\rho^i-h)\right\} \quad (25)$$

uniformly for $\rho\in\Gamma$.

The stated convergence is *uniform* in ρ on compact subsets of I_R. This allows us to take $M(\varepsilon,\rho)$ as our measure of periodic tuning, compute the scale $\rho_0(h)$ for which the transition rate is optimal, and define the *stochastic resonance point* as the eventually existing limit of $\rho_0(h)$ as $h\to 0$. This notion of quality has the big advantage of being robust for the passage from the two state Markov chain to the diffusion. So the following final robustness result holds true.

Theorem 12. *The resonance points of the diffusion $X^{\varepsilon,T}$ with periodic potential U and of the Markov chain $Y^{\varepsilon,T}$ with exponential transition rate functions $D_{\pm 1}$ coincide.*

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