WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 1

Exercise 1.
Consider the category $\mathcal{A}_\mathbb{Z}$ of $\mathbb{Z}$-**graded abelian groups**, the category $\mathcal{C}$ of **chain complexes** and the homotopy category $\mathcal{H}$, defined as follows:

- Objects of $\mathcal{A}_\mathbb{Z}$ are $\mathbb{Z}$-graded abelian groups, i.e. $G_* = \oplus_{n\in\mathbb{Z}} G_n$, where G_n is an abelian group, and morphisms from G_* to H_* are group homomorphisms $\phi\colon G_* \to H_*$ satisfying $\phi(G_n) \subset H_n$ for every $n \in \mathbb{Z}$.
- Objects of $\mathcal{C}$ are chain complexes (C_*, ∂) and morphisms from (C_*, ∂_C) to (D_*, ∂_D) are chain maps $\Phi\colon (C_*, \partial_C) \to (D_*, \partial_D)$.
- Objects of $\mathcal{H}$ are topolgical spaces and morphisms from X to Y are homotopy classes of continuous maps $X \to Y$.

(a) Verify that these define categories.

(b) The homology of a chain complex defines a functor H_* from $\mathcal{C}$ to $\mathcal{A}_\mathbb{Z}$.

(c) Let F be some functor from $\mathcal{H}$ to the category of groups $\mathcal{G}$ such that $F(D^n) = 0$ and $F(S^{n-1}) \neq 0$. Conclude from the existence of F the Brouwer fix point theorem.

Exercise 2.
Let (X, x_0) be a pointed topological space. We denote by $\pi_k(X, x_0)$ the homotopy classes of maps $(I^k, \partial I^k) \to (X, x_0)$.

(a) Complete the proof of Theorem 2.1 from the lecture, i.e. verify that $f * g$ as defined in the lecture really defines a group structure on $\pi_k(X, x_0)$.

Extra task: Describe the same group structure on the homotopy classes of maps $(S^k, N) \to (X, x_0)$.

(b) Prove that $\pi_k(X, x_0)$ is abelian (for $k \geq 2$) by explicitly writing down a homotopy between $f * g$ and $g * f$.

(c) Let γ be a path from x_0 to x_1 in X. Verify that the map $\gamma_\#$ as defined in the lecture induces an isomorphism from $\pi_k(X, x_0)$ to $\pi_k(X, x_1)$.

(d) The homotopy groups π_k define a covariant functor from the category $\mathcal{H}$ of homotopy classes of pointed topological spaces to the category $\mathcal{G}$ of groups.

Exercise 3.

(a) A map $f\colon (S^k, N) \to (X, x_0)$ extends to a map $F\colon D^{k+1} \to X$ if and only if $[f] = 0$ in $\pi_k(X, x_0)$.

Extra task: Is there a similar statement for relative homotopy groups?

(b) Prove the following generalization to higher dimensions of a special case of the Seifert and van Kampen theorem:
Let X be topological space with $X = U \cup V$ such that U and V are open and $(n-1)$-connected and $U \cap V$ is n-connected. Then

$$\pi_n(X) \cong \pi_n(U) \times \pi_n(V).$$

Extra task: Can there be a generalization of the general Seifert and van Kampen theorem to higher dimensions?

This sheet will be discussed on Wednesday 23.10. and should be solved by then.