Topology II

Exercise sheet 11

Exercise 1.

(a) $\mathrm{Hom}(\mathbb{Z}, G)$ is isomorphic to G for any abelian group G.

(b) $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m)$ is isomorphic to $\mathbb{Z}_{\gcd(n,m)}$.

(c) Compute $\mathrm{Hom}(A, B)$ for finitely generated abelian groups A and B.

(d) Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups and let G be another abelian group. If C is free abelian, then the dual sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \longrightarrow 0$$

is also exact.

(e) Does the dual sequence also split?

Exercise 2.

(a) Prove Lemma 6.5 from the lecture.

(b) Is Ext symmetric?

(c) Compute $\mathrm{Ext}(A, B)$ for finitely generated abelian groups A and B.

Exercise 3.
The evaluation map induces an isomorphism

$$\mathrm{ev}\colon H^k(X; \mathbb{R}) \longrightarrow \mathrm{Hom}_{\mathbb{R}}\big(H_k(X; \mathbb{R}), \mathbb{R}\big).$$

Exercise 4.
Compute the cohomology groups with $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$ and $\mathbb{R}$ coefficients of $\mathbb{R}P^n$, $\mathbb{R}P^\infty$ and all closed surfaces.

p.t.o.

Bonus exercise 1.

(a) The short exact sequence in the universal coefficient theorem for cohomology is natural.

(b) The short exact sequence in the universal coefficient theorem for cohomology is split.

(c) Show that the splitting of the short exact sequence in the universal coefficient theorem for cohomology cannot be natural.
Hint: This can be proven with similar methodes as in Exercise 1 from Sheet 10.

Bonus exercise 2.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.

(b) Conclude that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.

(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.
Hint: For this you will need the axiom of choice.

(d) Conclude that there exist topological spaces X and Y such that $H^k(X)$ is isomorphic to $H^k(Y)$ for all k, but such that $H_1(X)$ and $H_1(Y)$ are **not** isomorphic. In particular, the roles of homology and cohomology in Corollary 6.6 are not symmetric.
Hint: In [J. Wiegold, *Ext(Q, Z) is the additive group of real numbers*, Bull. Aust. Math. Soc. **1** (1969), 341–343] it is proven that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is isomorphic to $\mathbb{R}$. Use this (without proof) together with the universal coefficient theorem for cohomology and Exercise 3 from Sheet 7 to do the exercise.

This sheet will be discussed on Friday 17.1. and should be solved by then.