WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 12

Exercise 1.
Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle, $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with $\mathbb{Z}_2$-coefficients, with $\mathbb{Z}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 2.

(a) Compute the cohomology groups with arbitrary coeffcients of S^n in two ways:

- via the long exact sequence of a pair in cohomology, and
- via the Mayer–Vietoris sequence for cohomology.

(b) Compute the cohomology groups of all closed surfaces via the Mayer–Vietoris sequence for cohomology.

Exercise 3.

(a) Let $A \subset X$ be a closed subspace that is a deformation retract of some open neighborhood U. Then $H^k(X, A; G)$ is isomorphic to $\widetilde{H}^k(X/A; G)$ induced by the projection map $X \to X/A$.

(b) If A is a retract of X, then $H^k(X; G)$ is isomorphic to $H^k(A; G) \oplus H^k(X, A; G)$.

Exercise 4.
Let M and N be closed oriented n-manifolds and $f\colon M \to N$ a map. Then the induced map on cohomology $f^*\colon H^n(N; G) \to H^n(M; G)$ is the multiplication by $\deg(f)$.

Bonus exercise 1.
Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_1 is free abelian. Show that there exists a connected CW-complex X such that for any $k \in \mathbb{N}$ we have $H^k(X) \cong A_k$.

Remark: In contrast to homology groups, not every sequence of abelian groups can occur as cohomology groups of spaces. In [D. Kan and G. Whitehead, *On the realizability of singular cohomology groups,* Proc. Amer. Math. Soc. **12** (1961), 24–25] it is shown that there is no space X such that $H^k(X) = 0$ and $H^{k+1}(X) \cong \mathbb{Q}$.
It is unknown if $\mathbb{Q}$ can occur as cohomology group of a topological space at all.

This sheet will be discussed on Friday 24.1. and should be solved by then.