Topology II

Exercise sheet 13

Exercise 1.
Let X and Y be connected CW-complexes. We denote by $p_X\colon X \vee Y \to X$ and $p_Y\colon X \vee Y \to Y$ the projection maps.

(a) Show that $p_X^* \oplus p_Y^*\colon H^k(X;R) \oplus H^k(Y;R) \to H^k(X \vee Y;R)$ is an isomorphism for all $k \in \mathbb{N}$.

(b) The cup product $p_X^*(\alpha) \cup p_Y^*(\beta)$ is vanishing for all α and β of non-trivial degree.

(c) Compute the cup product on the cohomology $H^*(\Sigma_2)$ of the genus 2 surface Σ_2.
Hint: Consider maps $\Sigma_2 \to T^2$ and $\Sigma_2 \to T^2 \vee T^2$ and use the calculation of the cup product of T^2 from the lecture.

Bonus: What is the cup product of a general genus-g surface Σ_g?

Exercise 2.
We consider the ideal I in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ generated by x_i^2 and $x_i x_j + x_j x_i$ for all $i, j = 1, \ldots, n$. The **exterior algebra** $\Lambda[x_1, \ldots, x_n]$ is defined to be $\mathbb{Z}[x_1, \ldots, x_n]/I$, where the product is usually denote by $\wedge$ and $\deg(x_j) := 1$.

(a) $\Lambda[x_1, \ldots, x_n]$ is a free abelian group of rank 2^n where a basis is given by

$$\left\{ x_{k_1} \wedge \cdots \wedge x_{k_l} \,\middle|\, k_1 < \cdots < k_l \right\}.$$

(b) $\Lambda[x_1, \ldots, x_n]$ is isomorphic to $\Lambda[x_1, \ldots, x_{n-1}] \otimes \mathbb{Z}[x_n]/(x_n^2)$, where $\deg(x_n) := 1$.

(c) The cohomology ring of the n-torus $H^*(T^n)$ is isomorphic to $\Lambda[x_1, \ldots, x_n]$.

(d) Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$

by computing the Euler characteristic $\chi(T^n)$ via the alternating ranks of its cohomology groups (using the universal coeffcient theorem) and directly via a cell structure of T^n.

Exercise 3.

(a) Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have isomorphic homology and cohomology groups but different cohomology rings and thus are not homotopy equivalent.

(b) Use the cup product to show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map on first cohomology with $\mathbb{Z}_2$-coefficients if $n > m$.

(c) Deduce from (b) the Borsuk–Ulam theorem.

Exercise 4.

(a) For every integer $k \in \mathbb{Z}$ there exists a map $T^2 \to T^2$ of degree k.
Hint: In Exercise 2 from Sheet 7 we have constructed maps $S^n \to S^n$ of arbitrary degree.

(b) Now we consider a general genus g surface Σ_g with $g \geq 2$. Construct maps $\Sigma_g \to \Sigma_g$ with degree 0, 1 and -1.

(c) Any map $\Sigma_g \to \Sigma_g$ of non-vanishing degree 0 induces a surjection on fundamental groups.
Hint: Lift the map to a suitable covering of Σ_g, deduce from the non-vanishing of the degree that this covering has to be finite and use the behavior of the Euler characteristic under finite coverings.

(d) Deduce that any map $\Sigma_2 \to \Sigma_2$ has degree 0, 1 or -1.
Hint: Use (c) together with the Hurewicz homomorphism and the cup product structure of Σ_2 from Exercise 1 (c).

Bonus: Show the statement from part (d) for arbitrary genus $g \geq 2$ using the bonus part from Exercise 1.

Bonus exercise.
Let X be a CW-complex. The map

$$H_k(X) \times H_l(X) \xrightarrow{\times} H_{k+l}(X \times X) \xrightarrow{\mathrm{pr}^*} H_{k+l}(X)$$

is trivial, where $\times$ denotes the cross product and pr denotes the projection to one of the X-factors.

This sheet will be discussed on Friday 31.1. and should be solved by then.