WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 15

Exercise 1.
Let $Q\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be a symmetric bilinear form and let e_i be the standard generators of $\mathbb{Z}^n$.

(a) Q is non-singular if and only if its representing matrix $\big(Q(e_i, e_j)\big)_{1\leq i,j\leq n}$ is invertible over $\mathbb{Z}$ (which is equivalent to having determinant ± 1).

(b) Describe the matrices (in a suitable basis) of the intersection forms of $\pm\mathbb{C}P^2$, $S^2\times S^2$ and compute its ranks, signatures, parities and definiteness.

(c) Show that E_8 represents a non-singular symmetric positive definite bilinear form and compute its rank, signature and parity.

(d) Show that the intersection forms H and $-H$ are isomorphic.

(e) Show that the intersection forms H and $[+1]\oplus[-1]$ are isomorphic over $\mathbb{R}$ but not over $\mathbb{Z}$ and conclude that $S^2\times S^2$ is not homeomorphic to $-\mathbb{C}P^2\#\mathbb{C}P^2$, where $-\mathbb{C}P^2$ denotes $\mathbb{C}P^2$ with opposite orientation.

Exercise 2.

(a) Use Corollary 8.3 and the intersection form to compute the ring structure of $\mathbb{C}P^n$.

(b) Show that $\mathbb{C}P^{2m}$ admits no orientation reversing diffeomorphism.

(c) Show that any map $\mathbb{C}P^{2m}\to\mathbb{C}P^{2m}$ has a fixed point.
Hint: Use the Lefschetz fixed point theorem.

Bonus: What can you say about maps $\mathbb{C}P^{2m+1}\to\mathbb{C}P^{2m+1}$?

Exercise 3.
Show that the $\mathbb{Z}_2$-intersection form completely classifies closed surfaces.

Exercise 4.
Show that $S^2\vee S^4$ is not homotopy equivalent to a manifold.

This sheet will be discussed on Friday 14.2. and should be solved by then.