WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 2

Exercise 1.
Let X be a path-connected space and denote by CX its cone. Show that

$$\pi_k(CX, X) \cong \pi_{k-1}(X)$$

and construct for a given finitely presented group G a pair of path-connected spaces (Y, A) with $\pi_2(Y, A) \cong G$.

Exercise 2.

(a) The Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$ carries the structure of an S^1-bundle over S^1.

(b) We identify S^{2n+1} with the unit sphere in $\mathbb{C}^{n+1}$. The map

$$p\colon S^{2n+1} \longrightarrow \mathbb{C}P^n$$
$$(z_0, \cdots, z_n) \longrightarrow [z_0 : \cdots : z_n]$$

is an S^1-bundle.

(c) We denote by $\mathbb{H}$ the quaternions and define the **quaternionic projective spaces** $\mathbb{H}P^n$ as

$$\mathbb{H}P^n := \big(\mathbb{H}^{n+1} \setminus \{0\}\big)/_\sim,$$

where $u \sim v$ if and only if there exists an $h \in \mathbb{H} \setminus \{0\}$ such that $v = hw$. Verify that $\mathbb{H}P^n$ is a well-defined closed oriented manifold of dimension $4n$ and show that $\mathbb{H}P^1$ is homeomorphic to S^4.

(d) We identify S^{4n+3} with the unit sphere in $\mathbb{H}^{n+1}$. The map

$$p\colon S^{4n+1} \longrightarrow \mathbb{H}P^n$$
$$(h_0, \cdots, h_n) \longrightarrow [h_0 : \cdots : h_n]$$

is an S^3-bundle.

(e) What conclusion do we get from the above bundles about the homotopy groups of these spaces?

Exercise 3.
Let $X_1 \subset X_2 \subset X_3 \subset \cdots$ be an infinite sequence of inclusions of topological spaces. We define the limit

$$X_\infty := \varinjlim X_i := \bigcup_{i \in \mathbb{N}} X_i,$$

where a set U in X_∞ is called open if $U \cap X_i$ is open in X_i for all $i \in \mathbb{N}$.

If we apply the above construction to the sequence $S^0 \subset S^1 \subset S^2 \subset \cdots$ we get the space S^∞ and from the sequence $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots$ we get the spaces and $\mathbb{C}P^\infty$.

(a) $\pi_k(S^\infty) = 0$ for all $k \geq 1$.

(b) Define an S^1-bundle $p\colon S^\infty \to \mathbb{C}P^\infty$ in analogy to Exercise 2(b).

(c) Compute from the associated long exact sequence the homotopy groups of $\mathbb{C}P^\infty$ and conclude that S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups.

(d) Let Σ_g denote the closed oriented surface of genus $g \geq 2$. Compute all homotopy groups of Σ_g by constructing its universal cover.
Hint: Write Σ_g as a $4g$-gon Q with edges appropriately identified. Then construct the universal cover $\widetilde{\Sigma_g}$ by a limit of nested disks $D_1 \subset D_2 \subset D_3 \subset \cdots$, where D_1 is a single copy of Q and D_i is constructed from D_{i-1} by attaching additional copies of Q to D_{i-1} appropriately.

This sheet will be discussed on Friday 25.10. and should be solved by then.