WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 4

Exercise 1.
Prove Theorem 3.7 (or recall its proof from last semesters course) and deduce from it Corollary 3.8.

Exercise 2.
Prove the theorem of Mayer–Vietoris for singular homology:
Let $U, V \subset X$ be such that $X = \mathring{U} \cup \mathring{V}$. Then there exists a long exact sequence of the form

$$\cdots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(X) \to H_{k-1}(U \cap V) \to \cdots .$$

Hint: Use Theorem 3.7 and Lemma 3.15 from the lecture.

Exercise 3.

(a) Reformulate Corollary 3.10 using reduced homology groups.

(b) Compute again the homology groups of spheres by using Theorem 3.13.

(c) Show that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.
Hint: Consider $\mathbb{R}^n \setminus \{0\}$ or the one-point compactification of $\mathbb{R}^n$.

(d) Compute the homology groups of all spheres and all closed surfaces using the Mayer–Vietoris-Sequence from Exercise 2.

Bonus: Is there a way to compute the homology groups of all closed surfaces using Theorem 3.13?

Exercise 4.
Let $x_0 \in X$ be a point.

(a) $\widetilde{H}_k(X) \cong H_k(X)$ for $k \geq 1$.

(b) $\widetilde{H}_0(X) \cong \mathbb{Z}^{n-1}$, where n is the number of path components of X.

(c) $\widetilde{H}_k(X) \cong H_k(X, \{x_0\})$.

Bonus exercise.
Construct a pair of spaces (X, A) such that $\widetilde{H}_k(X/A)$ is **not** isomorphic to $H_k(X, A)$.

This sheet will be discussed on Friday 15.11. and should be solved by then.