Topology II

Exercise sheet 6

Exercise 1.

(a) $\mathbb{R}^\infty$ is homeomorphic to $\mathbb{C}^\infty$.

(b) Choose an explicit CW-structure on S^∞, describe the corresponding cellular chain complex and compute the cellular homology of S^∞.

(c) S^∞ is contractible.

Exercise 2.
Compute the degree of the constant map $\mathrm{c}\colon S^n \to S^n$, the identity $\mathrm{Id}\colon S^n \to S^n$ and the antipodal map $\text{-Id}\colon S^n \to S^n$.

Exercise 3.
Compute the cellular homology groups of all compact surfaces.

Exercise 4.
We denote by F a 2-torus with two open 2-balls removed.
The **surface of infinite genus** Σ_∞ is defined to be the direct limit of the nested sequence of topological spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ where X_0 is a copy of $S^1 \times I$ and X_{i+1} is obtained from X_i by attaching a copy of F to each boundary component of X_i.
The **Loch Ness monster surface** L_∞ is the non-compact surface obtained as direct limit of $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ where Y_0 is a 2-disk and Y_{i+1} is obtained from Y_i by attaching a copy of F to its boundary.

(a) Make sketches of Σ_∞ and L_∞. What has L_∞ to do with the Loch Ness monster?

(b) Describe CW-structures on L_∞ and Σ_∞ and compute its cellular homology groups.

(c) Are L_∞ and Σ_∞ homeomorphic?

Bonus exercise.
Show that the simplicial homology groups of a triangulizable space X are isomorphic to its cellular homology groups.
Hint: Let T be a triangulation of a topological space X. In the lecture we defined a CW-structure on X coming from T. Consider the corresponding simplicial- and cellular chain complexes.

Bonus exercise.
Let M be a smooth compact n-manifold. Then we have seen in lecture that there exists a triangulation of M such that ∂M is a subcomplex. We denote by $H_n^{simp}(M, \partial M)$ its n-th simplicial relative homology group. Show that

$$H_n^{simp}(M, \partial M) \cong \begin{cases} \mathbb{Z}\,; & \text{if } M \text{ is orientable,} \\ 0\,; & \text{if } M \text{ is not orientable.} \end{cases}$$

Hint: Adapt last semesters proof for closed manifolds.

This sheet will be discussed on Friday 29.11. and should be solved by then.