Topology II

Exercise sheet 7

Exercise 1.
We define the Euler characteristic $\chi(X)$ of a finite CW-complex X of dimension n to be

$$\chi(X) := \sum_{k=0}^{n} (-1)^k |I_k|,$$

where $|I_k|$ denotes the number of k-cells in X.

(a) Compute the Euler characteristic for your favorite CW-structure of S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and Σ_g.

(b) Show that the Euler characteristic of a CW-complex only depends on the homotopy type of X and not on the particular CW-structure.
Hint: Relate the Euler characteristic of a CW-complex to its cellular homology groups.

Bonus: Conclude that the Euler characteristic of a CW-complex agrees with the Euler characteristic we defined last semester.

(c) Let X and Y be finite CW-complexes. Find a way to compute $\chi(X \times Y)$ from the Euler characteristics of X and Y.

Exercise 2.
Construct, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, a map $f\colon S^n \to S^n$ of degree k.
Hint: Construct from a given map $S^{n-1} \to S^{n-1}$ a map $S^n \to S^n$ of the same degree by identifying S^n with the suspension ΣS^{n-1}.

Exercise 3.
Let $G_1, G_2, \dots$ be a (possible infinite) sequence of (not necessarily finitely presented) abelian groups. Construct a CW-complex X with reduced homology groups

$$\widetilde{H}_k(X) \cong \begin{cases} G_k\,; & \text{if } k \in \mathbb{N}, \\ 0\,; & \text{else.} \end{cases}$$

Hint: It may be helpful to start with sequences of finitely presented groups where only finitely many groups are non-trivial.
Bonus: Is such a space unique up to homotopy?

This sheet will be discussed on Friday 6.12. and should be solved by then.