WS 2019/2020 Marc Kegel

Topology II

Exercise sheet 9

Exercise 1.

(a) Let A be a finitely generated abelian group, i.e. A is of the form $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$ for natural numbers a_i. For B isomorphic to $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ we have

$$A \otimes B \cong B^r.$$

(b) For finitely generated abelian groups A and B we have

$$\operatorname{rk}(A \otimes B) = \operatorname{rk}(A) \cdot \operatorname{rk}(B).$$

(c) Let R be a commutative ring. Then

$$\begin{aligned} A &\longmapsto A \otimes R \\ (f\colon A \to B) &\longmapsto (f \otimes \mathrm{id}\colon A \otimes R \to B \otimes R) \end{aligned}$$

defines a functor from the category of abelian groups to the category of R-modules.
Hint: If you are unfamiliar with the notions of rings and modules, it may be helpful to first consider the case that R is a field. (Then R-modules are just the R-vector spaces.)

Exercise 2.
Let $\mathbb{F}$ be an arbitrary field and X a finite CW-complex of dimension n. Then the Euler characteristic of X is given by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} H_k(X, \mathbb{F}).$$

Exercise 3.
In the lecture we have seen four different ways to compute homology groups with coefficients:

- via the Mayer–Vietoris sequence,
- directly from the definition and a CW-structure,
- with the Bockstein homomorphism or
- using the universal coefficient theorem.

Compute the homology groups of $\mathbb{R}P^n$ and the Klein bottle with $\mathbb{Q}$- and $\mathbb{Z}_2$-coefficients using as many of the above methods as possible.
Bonus: Do the same for $\mathbb{Z}_p$- and $\mathbb{R}$-coefficients.

Exercise 4.

(a) A short exact sequence

$$0 \to B \to C \to D \to 0$$

of abelian groups induces an exact sequence of the form

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \oplus B \to A \oplus C \to A \oplus D \to 0.$$

(b) Prove Lemma 5.5 from the lecture.

Bonus exercise 1.
Classify finitely generated abelian groups, i.e. prove that any finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}$$

for natural numbers a_i.

Bonus exercise 2.

(a) Prove that the Hurewitcz homomorphism is a natural homomorphism.

(b) Let X be an $(n-1)$-connected space. Show that

$$\widetilde{H}_k(X) = 0$$

for all $k < n$.
Hint: Use the strategy from the proof of Theorem 4.17 from the lecture.

This sheet will be discussed on Friday 20.12. and should be solved by then.