Exercise 1.

(a) Let $A$ be a finitely generated abelian group, i.e. $A$ is of the form $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$, for natural numbers $a_i$. For $B$ isomorphic to $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ we have

$$A \otimes B \cong B^r.$$

(b) For finitely generated abelian groups $A$ and $B$ we have

$$\text{rk}(A \otimes B) = \text{rk}(A) \cdot \text{rk}(B).$$

(c) Let $R$ be a commutative ring. Then

$$A \mapsto A \otimes R$$

$$(f : A \to B) \mapsto (f \otimes \text{id} : A \otimes R \to B \otimes R)$$

defines a functor from the category of abelian groups to the category of $R$-modules.

*Hint:* If you are unfamiliar with the notions of rings and modules, it may be helpful to first consider the case that $R$ is a field. (Then $R$-modules are just the $R$-vector spaces.)

Exercise 2.

Let $\mathbb{F}$ be an arbitrary field and $X$ a finite $CW$-complex of dimension $n$. Then the Euler characteristic of $X$ is given by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} H_k(X, \mathbb{F}).$$

Exercise 3.

In the lecture we have seen four different ways to compute homology groups with coefficients:

- via the Mayer–Vietoris sequence,
- directly from the definition and a $CW$-structure,
- with the Bockstein homomorphism or
- using the universal coefficient theorem.

Compute the homology groups of $\mathbb{R}P^n$ and the Klein bottle with $\mathbb{Q}$- and $\mathbb{Z}_2$-coefficients using as many of the above methods as possible.

**Bonus:** Do the same for $\mathbb{Z}_p$- and $\mathbb{R}$-coefficients.

p.t.o.
Exercise 4.

(a) A short exact sequence
\[ 0 \to B \to C \to D \to 0 \]
of abelian groups induces an exact sequence of the form
\[ 0 \to \text{Tor}(A, B) \to \text{Tor}(A, C) \to \text{Tor}(A, D) \to A \oplus B \to A \oplus C \to A \oplus D \to 0. \]

(b) Prove Lemma 5.5 from the lecture.

Bonus exercise 1.
Classify finitely generated abelian groups, i.e. prove that any finitely generated abelian group is isomorphic to a group of the form
\[ \mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i} \]
for natural numbers \( a_i \).

Bonus exercise 2.

(a) Prove that the Hurewitz homomorphism is a natural homomorphism.

(b) Let \( X \) be an \((n - 1)\)-connected space. Show that
\[ \widetilde{H}_k(X) = 0 \]
for all \( k < n \).

\textit{Hint:} Use the strategy from the proof of Theorem 4.17 from the lecture.

This sheet will be discussed on Friday 20.12. and should be solved by then.