

Topology II

Exercise sheet 10

Exercise 1.

- (a) Prove Lemma 6.5 from the lecture.
- (b) Is Ext symmetric?
- (c) Compute $\text{Ext}(A, B)$ for finitely generated abelian groups A and B .

Exercise 2.

Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle, $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with \mathbb{Z}_2 -coefficients, with \mathbb{Z}_3 -coefficients and with \mathbb{Z} -coefficients.

Exercise 3.

- (a) Compute the cohomology groups with arbitrary coefficients of S^n in two ways:
 - via the long exact sequence of a pair in cohomology, and
 - via the Mayer–Vietoris sequence for cohomology.
- (b) Compute the cohomology groups of all closed surfaces via the Mayer–Vietoris sequence for cohomology.

Exercise 4.

- (a) Let $A \subset X$ be a closed subspace that is a deformation retract of some open neighborhood U . Then $H^k(X, A; G)$ is isomorphic to $\tilde{H}^k(X/A; G)$ induced by the projection map $X \rightarrow X/A$.
- (b) If A is a retract of X , then $H^k(X; G)$ is isomorphic to $H^k(A; G) \oplus H^k(X, A; G)$.

Bonus exercise 1.

- (a) Let G and H be \mathbb{Q} -vector spaces. Let $\varphi: G \rightarrow H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of \mathbb{Q} -vector spaces.
- (b) Conclude that the abelian groups \mathbb{Q} and \mathbb{Q}^2 are not isomorphic.
- (c) Show that the abelian groups \mathbb{R} and \mathbb{R}^2 are isomorphic.
Hint: For this you will need the axiom of choice.
- (d) Conclude that there exist topological spaces X and Y such that $H^k(X)$ is isomorphic to $H^k(Y)$ for all k , but such that $H_1(X)$ and $H_1(Y)$ are **not** isomorphic. In particular, the roles of homology and cohomology in Corollary 6.6 are not symmetric.
Hint: In [J. WIEGOLD, *Ext(Q, Z) is the additive group of real numbers*, Bull. Aust. Math. Soc. **1** (1969), 341–343] it is proven that $\text{Ext}(\mathbb{Q}, \mathbb{Z})$ is isomorphic to \mathbb{R} . Use this (without proof) together with the universal coefficient theorem for cohomology and Exercise 4 from Sheet 7 to do the exercise.