WS 2020/2021 Marc Kegel

Topology II

Exercise sheet 11

Exercise 1.
Let X and Y be connected CW-complexes. We denote by $p_X\colon X \vee Y \to X$ and $p_Y\colon X \vee Y \to Y$ the projection maps.

(a) Show that $p_X^* \oplus p_Y^*\colon H^k(X;R) \oplus H^k(Y;R) \to H^k(X \vee Y;R)$ is an isomorphism for all $k \in \mathbb{N}$.

(b) The cup product $p_X^*(\alpha) \cup p_Y^*(\beta)$ is vanishing for all α and β of non-trivial degree.

(c) Compute the cup product on the cohomology $H^*(\Sigma_2)$ of the genus 2 surface Σ_2.
Hint: Consider maps $\Sigma_2 \to T^2$ and $\Sigma_2 \to T^2 \vee T^2$ and use the calculation of the cup product of T^2 from the lecture.

Bonus: What is the cup product of a general genus-g surface Σ_g?

Exercise 2.
We consider the ideal I in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ generated by x_i^2 and $x_i x_j + x_j x_i$ for all $i, j = 1, \dots, n$. The **exterior algebra** $\Lambda[x_1, \dots, x_n]$ is defined to be $\mathbb{Z}[x_1, \dots, x_n]/I$, where the product is usually denote by $\wedge$ and $\deg(x_j) := 1$.

(a) $\Lambda[x_1, \dots, x_n]$ is a free abelian group of rank 2^n where a basis is given by

$$\left\{ x_{k_1} \wedge \dots \wedge x_{k_l} \middle| k_1 < \dots < k_l \right\}.$$

(b) $\Lambda[x_1, \dots, x_n]$ is isomorphic to $\Lambda[x_1, \dots, x_{n-1}] \otimes \mathbb{Z}[x_n]/(x_n^2)$, where $\deg(x_n) := 1$.

(c) The cohomology ring of the n-torus $H^*(T^n)$ is isomorphic to $\Lambda[x_1, \dots, x_n]$.

(d) Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$

by computing the Euler characteristic $\chi(T^n)$ via the alternating ranks of its cohomology groups (using the universal coefficient theorem) and directly via a cell structure of T^n.

Exercise 3.
Let M and N be closed oriented n-manifolds and $f\colon M \to N$ a map. Then the induced map on cohomology $f^*\colon H^n(N;G) \to H^n(M;G)$ is the multiplication by $\deg(f)$.

Exercise 4.

(a) Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have isomorphic homology and cohomology groups but different cohomology rings and thus are not homotopy equivalent.

(b) Use the cup product to show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map on first cohomology with $\mathbb{Z}_2$-coefficients if $n > m$.

(c) Deduce from (b) the Borsuk–Ulam theorem.

Bonus exercise 1.
Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_1 is free abelian. Show that there exists a connected CW-complex X such that for any $k \in \mathbb{N}$ we have $H^k(X) \cong A_k$.

Remark: In contrast to homology groups, not every sequence of abelian groups can occur as cohomology groups of spaces. In [D. Kan and G. Whitehead, *On the realizability of singular cohomology groups,* Proc. Amer. Math. Soc. **12** (1961), 24–25] it is shown that there is no space X such that $H^k(X) = 0$ and $H^{k+1}(X) \cong \mathbb{Q}$.
It is unknown if $\mathbb{Q}$ can occur as cohomology group of a topological space at all.

Bonus exercise 2.
Let X be a CW-complex. The map

$$H_k(X) \times H_l(X) \xrightarrow{\times} H_{k+l}(X \times X) \xrightarrow{\mathrm{pr}^*} H_{k+l}(X)$$

is trivial, where $\times$ denotes the cross product and pr denotes the projection to one of the X-factors.

This sheet will be discussed on Friday 5.2. and should be solved by then.