

Topology II

Exercise sheet 3

Exercise 1.

Let X be the topological space consisting of a single point. Compute all homology groups of X directly from the definitions. For which other spaces can we compute all homology groups?

Exercise 2.

Find a way to relate the homology group of a topological space X to the homology groups of its path-connected components.

Exercise 3.

(a) The Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$ carries the structure of an S^1 -bundle over S^1 .

(b) We identify S^{2n+1} with the unit sphere in \mathbb{C}^{n+1} . The map

$$\begin{aligned} p: S^{2n+1} &\longrightarrow \mathbb{C}P^n \\ (z_0, \dots, z_n) &\longrightarrow [z_0 : \dots : z_n] \end{aligned}$$

is an S^1 -bundle.

(c) We denote by \mathbb{H} the quaternions and define the **quaternionic projective spaces** $\mathbb{H}P^n$ as

$$\mathbb{H}P^n := (\mathbb{H}^{n+1} \setminus \{0\}) / \sim,$$

where $u \sim v$ if and only if there exists an $h \in \mathbb{H} \setminus \{0\}$ such that $v = hw$. Verify that $\mathbb{H}P^n$ is a well-defined closed oriented manifold of dimension $4n$ and show that $\mathbb{H}P^1$ is homeomorphic to S^4 .

(d) We identify S^{4n+3} with the unit sphere in \mathbb{H}^{n+1} . The map

$$\begin{aligned} p: S^{4n+3} &\longrightarrow \mathbb{H}P^n \\ (h_0, \dots, h_n) &\longrightarrow [h_0 : \dots : h_n] \end{aligned}$$

is an S^3 -bundle.

(e) What conclusion do we get from the above bundles about the homotopy groups of these spaces?

Exercise 4.

Let $X_1 \subset X_2 \subset X_3 \subset \dots$ be an infinite sequence of inclusions of topological spaces. We define the limit

$$X_\infty := \varinjlim X_i := \bigcup_{i \in \mathbb{N}} X_i,$$

where a set U in X_∞ is called open if $U \cap X_i$ is open in X_i for all $i \in \mathbb{N}$.

If we apply the above construction to the sequence $S^0 \subset S^1 \subset S^2 \subset \dots$ we get the space S^∞ and from the sequence $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \dots$ we get the spaces and $\mathbb{C}P^\infty$.

- (a) $\pi_k(S^\infty) = 0$ for all $k \geq 1$.
- (b) Define an S^1 -bundle $p: S^\infty \rightarrow \mathbb{C}P^\infty$ in analogy to Exercise 2(b).
- (c) Compute from the associated long exact sequence the homotopy groups of $\mathbb{C}P^\infty$ and conclude that S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups.