WS 2020/2021 Marc Kegel

Topology II

Exercise sheet 7

Exercise 1.

(a) Describe CW-decompositions of all closed surfaces and compute their cellular homology.

(b) The n-torus $T^n := S^1 \times \cdots S^1$ admits the structure of a CW-complex.

(c) Describe CW-complexes on $\Sigma_g \times \Sigma_h$.
Hint: Use (a) and the technique from (b).

Exercise 2.

(a) $\mathbb{R}^\infty$ is homeomorphic to $\mathbb{C}^\infty$.

(b) Choose an explicit CW-structure on S^∞, describe the corresponding cellular chain complex and compute the cellular homology of S^∞.

(c) S^∞ is contractible.

Exercise 3.
We denote by F a 2-torus with two open 2-balls removed.
The **surface of infinite genus** Σ_∞ is defined to be the direct limit of the nested sequence of topological spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ where X_0 is a copy of $S^1 \times I$ and X_{i+1} is obtained from X_i by attaching a copy of F to each boundary component of X_i.
The **Loch Ness monster surface** L_∞ is the non-compact surface obtained as direct limit of $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ where Y_0 is a 2-disk and Y_{i+1} is obtained from Y_i by attaching a copy of F to its boundary.

(a) Make sketches of Σ_∞ and L_∞. What has L_∞ to do with the Loch Ness monster?

(b) Describe CW-structures on L_∞ and Σ_∞ and compute its cellular homology groups.

(c) Are L_∞ and Σ_∞ homeomorphic?

Exercise 4.
Let $G_1, G_2, \ldots$ be a (possible infinite) sequence of (not necessarily finitely presented) abelian groups. Construct a CW-complex X with reduced homology groups

$$\widetilde{H}_k(X) \cong \begin{cases} G_k\,; & \text{if } k \in \mathbb{N}, \\ 0\,; & \text{else.} \end{cases}$$

Hint: It may be helpful to start with sequences of finitely presented groups where only finitely many groups are non-trivial.
Bonus: Is such a space unique up to homotopy?

Exercise 5.

We define the Euler characteristic $\chi(X)$ of a finite CW-complex X of dimension n to be

$$\chi(X) := \sum_{k=0}^{n} (-1)^k |I_k|,$$

where $|I_k|$ denotes the number of k-cells in X.

(a) Compute the Euler characteristic for your favorite CW-structure of S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and Σ_g.

(b) Show that the Euler characteristic of a CW-complex only depends on the homotopy type of X and not on the particular CW-structure.
Hint: Relate the Euler characteristic of a CW-complex to its cellular homology groups.

(c) Let K and L be CW-complexes that intersect in a common subcomplex $K \cap L$. Verify the gluing formula for the Euler characteristic:

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L).$$

(d) Let X and Y be finite CW-complexes. Find a way to compute $\chi(X \times Y)$ from the Euler characteristics of X and Y.

(e) The Euler characteristic together with the orientability is a complete invariant of closed surfaces.

(f) The 2-sphere admits a CW-decomposition with an arbitrary number of even cells, but no CW-decomposition with an odd number of cells.

Bonus exercise 1.

Let M be a smooth compact n-manifold. We define $\Delta(M)$ as the minimal number of n-simplices in a triangulation of M. We define similarly $c(M)$ as the minimal number of cells in a cell decomposition of M.

(a) Compute Δ und c for S^2, $\mathbb{R}P^2$ und T^2.

(b) What can you say about Δ and c for other surfaces?

Bonus exercise 2.

Show that the simplicial homology groups of a triangulizable space X are isomorphic to its cellular homology groups.
Hint: Let T be a triangulation of a topological space X. In the lecture we defined a CW-structure on X coming from T. Consider the corresponding simplicial- and cellular chain complexes.

Bonus exercise 3.

Let X be a finite connected CW-complex. How can the fundamental group of X be computed?
Bonus: What can be said if X is **not** finite?

This sheet will be discussed on Friday 8.1. and should be solved by then.