WS 2020/2021 Marc Kegel

Topology II

Exercise sheet 8

Exercise 1.

(a) Let A be a finitely generated abelian group, i.e. A is of the form $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$ for natural numbers a_i. For B isomorphic to $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ we have

$$A \otimes B \cong B^r.$$

(b) For finitely generated abelian groups A and B we have

$$\mathrm{rk}(A \otimes B) = \mathrm{rk}(A) \cdot \mathrm{rk}(B).$$

(c) Let R be a commutative ring. Then

$$A \longmapsto A \otimes R$$
$$(f\colon A \to B) \longmapsto (f \otimes \mathrm{id}\colon A \otimes R \to B \otimes R)$$

defines a functor from the category of abelian groups to the category of R-modules.
Hint: If you are unfamiliar with the notions of rings and modules, it may be helpful to first consider the case that R is a field. (Then R-modules are just the R-vector spaces.)

Exercise 2.
Let $\mathbb{F}$ be an arbitrary field and X a finite CW-complex of dimension n. Then the Euler characteristic of X is given by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} H_k(X, \mathbb{F}).$$

Exercise 3.
In the lecture we have seen four different ways to compute homology groups with coefficients:

- via the Mayer–Vietoris sequence,
- directly from the definition and a CW-structure,
- with the Bockstein homomorphism or
- using the universal coefficient theorem.

Compute the homology groups of $\mathbb{R}P^n$ and the Klein bottle with $\mathbb{Q}$- and $\mathbb{Z}_2$-coefficients using as many of the above methods as possible.
Bonus: Do the same for $\mathbb{Z}_p$- and $\mathbb{R}$-coefficients.

p.t.o.

Exercise 4.

(a) A short exact sequence

$$0 \to B \to C \to D \to 0$$

of abelian groups induces an exact sequence of the form

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

(b) Prove Lemma 5.5 from the lecture.

Bonus exercise.
Classify finitely generated abelian groups, i.e. prove that any finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}$$

for natural numbers a_i.

Bonus exercise.

(a) Let X be a connected 1-dimensional CW-complex. Show $\pi_n(X) = 0$ for all $n \geq 2$.

(b) Compute all homotopy groups of surfaces Σ_g of genus $g \geq 1$ by applying Hurewitcz's theorem to its universal covering.

(c) Compute the second homotopy groups of $\mathbb{C}P^n$ and $S^1 \vee S^2$.

Bonus exercise.
A connected topological space X with only one non-vanishing homotopy group $\pi_n(X) \cong G$ is called **Eilenberg–MacLane space** $K(G, n)$.

(a) Construct an Eilenberg–MacLane space for arbitrary G and n (assuming G to be abelian if $n \geq 1$).
Hint: It may be helpful to have a look at Exercise 3 from Sheet 7.

(b) Let $G_1, G_2, \ldots$ be a (possible infinite) sequence of (not necessarily finitely presented) groups (abelian for $n \neq 1$). Construct a connected CW-complex X with homotopy groups

$$\pi_k(X) \cong G_k.$$

(c) When is such a space unique up to homotopy?

This sheet will be discussed on Friday 15.1. and should be solved by then.