WS 2020/2021 Marc Kegel

# Topology II

Exercise sheet 9

**Exercise 1.**
Compute the cohomology groups with $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$ and $\mathbb{R}$ coefficients of $\mathbb{R}P^n$, $\mathbb{R}P^\infty$ and all closed surfaces.

**Exercise 2.**
Show that the short exact sequence from the universal coefficient theorem does **not** split natural.
*Hint:* Consider the projection map

$$f\colon \mathbb{R}P^2 \cong D^2 \cup \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2/\mathbb{R}P^1 \cong S^2.$$

**Exercise 3.**

(a) Compute the homology of the $n$-torus $T^n := S^1 \times \cdots \times S^1$.

(b) Let $M$ and $N$ be closed **topological** manifolds. Show that $M \times N$ is orientable if and only if $M$ and $N$ are orientable.

**Exercise 4.**

(a) $\mathrm{Hom}(\mathbb{Z}, G)$ is isomorphic to $G$ for any abelian group $G$.

(b) $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m)$ is isomorphic to $\mathbb{Z}_{\gcd(n,m)}$.

(c) Compute $\mathrm{Hom}(A, B)$ for finitely generated abelian groups $A$ and $B$.

(d) Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups and let $G$ be another abelian group. If $C$ is free abelian, then the dual sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \longrightarrow 0$$

is also exact.

(e) Does the dual sequence also split?

**Bonus exercise 1.**
A map $f\colon X \to Y$ induces an isomorphism on homology with $\mathbb{Z}$-coefficients if and only if $f$ induces an isomorphism on homology with $\mathbb{Q}$-coeffcients and $\mathbb{Z}_p$-coeffcients for all primes $p$.

**Bonus exercise 2.**
$\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup of $A$.

This sheet will be discussed on Friday 22.1. and should be solved by then.