Exercise 3 (c)

This is the written version of the solution which I described in the problem session.

Want:- for $f : S^n \to \mathbb{R}^n$ $\exists$ $x \in S^n$ s.t. $f(x) = f(-x)$.

We proceed by contradiction. Suppose $\forall$ $x \in S^n$, $f(x) \neq f(-x)$. Consider the map

$$g : S^n \longrightarrow S^{n-1}$$
$$x \longmapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

which is well-defined as the denominator is never $= 0$.

Also, $g(-x) = -g(x)$.

$\therefore$ we have the following situation

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^{n-1} \\ p \downarrow & & \downarrow p \\ \mathbb{R}P^n & \xrightarrow{\tilde{g}} & \mathbb{R}P^{n-1} \end{array}$$

p is the 2-sheeted covering map.

If $\gamma \in \pi_1(\mathbb{RP}^n)$, i.e., γ is a loop in $\mathbb{RP}^n$ representing the generator of π_1 then it lifts to a path in S^n from a point x to its antipodal $-x$. The image of this path in S^n under g is a path from $g(x)$ to its antipode $g(-x) = -g(x)$. So the image of this path under p is a loop that is null-homotopic

$\Rightarrow$ we have $\tilde{g}_* : \pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$

which is nontrivial.

for $n = 1$, $\mathbb{RP}^0$ is a point But there are no nontrivial hom. from $\mathbb{Z} \to 0$, so we have a contradiction and hence Borsuk-Ulam holds.

for $n = 2$, $\tilde{g}_*$ gives a non-trivial hom. from $\mathbb{Z}_2 \to \mathbb{Z}$ which is again a contradiction.

$\therefore$ now assume $n > 2$ and so $\tilde{g}_*$ is an isomorphism of $\mathbb{Z}_2$.

Using the abelianization of π_1 as H_1 w/ the abelianization map being an isomorphism for path-connected $\mathbb{R}\mathbb{P}^n$ and abelian $\pi_1(X)$, we have

$$\begin{tikzcd}
\pi_1(\mathbb{R}\mathbb{P}^n) \arrow[r, "\tilde{g}_*", "\simeq"'] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_1(\mathbb{R}\mathbb{P}^{n-1}) \arrow[d, "ab", "\simeq"'] \\
H_1(\mathbb{R}\mathbb{P}^n) \arrow[r, "\tilde{g}_*"] & H_1(\mathbb{R}\mathbb{P}^{n-1})
\end{tikzcd}$$

$\tilde{g}_*$ i.e the bottom row is also an isomorphism.

Now apply the universal coeff. theorem to get

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{R}\mathbb{P}^{n-1}; \mathbb{Z}_2) \arrow[r] \arrow[d, "(\tilde{g}_*)^*"] & \mathrm{Hom}(H_1(\mathbb{R}\mathbb{P}^{n-1}); \mathbb{Z}_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(\mathbb{R}\mathbb{P}^n; \mathbb{Z}_2) \arrow[r] & \mathrm{Hom}(H_1(\mathbb{R}\mathbb{P}^n); \mathbb{Z}_2) \arrow[r] & 0
\end{tikzcd}$$

Ques:- why are these spaces 0?

$\therefore$ $(\tilde{g}_*)^*$ must be an isomorphism which contradicts part (b).

So we get Borsuk-Ulam theorem.

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