(4) X is path-connected, CX is the cone of X, i.e.

$$CX = X \times [0,1] \big/ X \times \{0\}$$

Also notice that any space X is a subspace of its cone via the identification $X \cong X \times \{1\} \subset CX$.

Thus we can talk about the relative homotopy groups $\pi_k(CX, X, x_0)$. We have the following long exact sequence for relative homotopy groups

$$\cdots \longrightarrow \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(CX, x_0) \xrightarrow{j_*} \pi_n(CX, X, x_0) \xrightarrow{\partial} \pi_{n-1}(X, x_0)$$

$$\xrightarrow{i_*} \pi_{n-1}(CX, x_0) \xrightarrow{j_*} \pi_{n-1}(CX, X, x_0) \longrightarrow \cdots \qquad \text{— (1)}$$

The cone CX is always contractible, i.e., it has the hom. type of a point.

(You can prove this as follows:-

Let $H: X \times [0,1] \times [0,1] \longrightarrow X \times [0,1]$ be defined by

$H((x,t), s) = (x, (1-s)t)$. Clearly H is continuous.

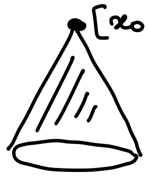
$H((x,0), s) = (x,0)$ and $H((x,t), 0) = (x,t)$
$H((x,t), 1) = (x,0)$

So we get an induced map $\tilde{H}: CX \times [0,1] \longrightarrow CX$ s.t.

$\tilde{H}([(x,t)], s) = [(x, (1-s)t)]$. Note that $\tilde{H}([(x,t)], 0)$

$= [(x,t)]$ and $\tilde{H}([(x,t)],1) = [(x,0)] = X \times \{0\}$

and hence $\tilde{H}$ is the required homotopy b/w id_{CX} to the point $X \times \{0\}$ ie CX. This should also be intuitively clear



$[x_0]$ as you are contracting CX to the point $[x_0]$.

$\because$ CX is contractible $\Rightarrow$ $\pi_n(CX, x_0) = 0$ and $\therefore$ ie ①

we have an isomorphism

$$\pi_k(CX, X) \cong \pi_{k-1}(X).$$

for the second part, we know from the first part that

$\pi_2(CX, X) \cong \pi_1(X)$. So all we need to do is to prove that given a finitely presented group G $\exists$ X w/ $\pi_1(X) \cong G$. But this is just an application of the van Kampen theorem.

Suppose $G = \langle g_\alpha \mid r_\beta \rangle$ where $\{g_\alpha\}_{\alpha \in I}$ are the generators and $\{r_\beta\}_{\beta \in I'}$ are the relations.

To construct X, we first take wedge of circles to get the generators of the group: $\bigvee_{\alpha \in I} S^1_\alpha$. To take care of the relations, we attach 2-cells e^2_β (a 2-cell is just the interior of a polygon) by the loops specified by words r_p. A similar discussion took place when we explicitly computed the fundamental groups of compact surfaces.

Bonus

We have the following comm. diagram w/ exact rows.

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\circlearrowright"] & A_2 \arrow[r, "i_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\circlearrowright"] & A_3 \arrow[r, "i_3"] \arrow[d, "f_3"] \arrow[dr, phantom, "\circlearrowright"] & A_4 \arrow[r, "i_4"] \arrow[d, "f_4"] \arrow[dr, phantom, "\circlearrowright"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "j_1"'] & B_2 \arrow[r, "j_2"'] & B_3 \arrow[r, "j_3"'] & B_4 \arrow[r, "j_4"'] & B_5
\end{tikzcd}$$

This exercise is an example of what is known as diagram chasing. Let me do some of the exercise.

want conditions on f_1, f_2, f_4, f_5 s.t. f_3 is

i) injective.

Let $a_3 \in A_3$ s.t. $a_3 \in \ker f_3$, so by commutativity

$$j_3 f_3(a_3) = f_4\, i_3(a_3) = 0$$

so if we want a relation b/w a_3 and $\ker i_3$ then we need f_4 to be injective. Thus if f_4 is injective then we get $i_3(a_3) = 0 \Rightarrow a_3 \in \ker i_3 \Rightarrow$ by exactness

$\exists\ a_2 \in A_2$ s.t. $i_2(a_2) = a_3$. —— (*)

There is only one thing we can do now in order to get f_i's into picture and that is to use commutativity. Using that we get.

$$f_3\, i_2(a_2) = f_3(a_3) = j_2\, f_2(a_2) = 0.$$

$\Rightarrow$ $f_2(a_2) \in \ker j_2$ and so again we can only do one thing and that is to use the exactness of the bottom row. So $\exists\ b_1 \in B_1$ s.t.

$$j_1(b_1) = f_2(a_2)$$

To get f_1 into picture, we can only use b_1 from the previous equation, so we assume that f_1 is surjective.

Thus $\exists\ a_1 \in A_1$ s.t. $b_1 = f_1(a_1)$

$$\Rightarrow \quad j_1\, f_1(a_1) = f_2(a_2)$$

But again using commutativity, we have

$$j_1 f_1(a_1) = f_2(a_2) = f_2\, i_1(a_1).$$

So we assume that f_2 is injective to get

$$a_2 = i_1(a_1)$$

But if $a_2 \in \operatorname{im} i_1 \Rightarrow$ by exactness, $a_2 \in \ker i_2$

$\Rightarrow$ $i_2(a_2) = 0$ and $\therefore$ from (*) $a_3 = i_2(a_2) = 0$

and hence f_3 is injective. Thus the conditions required are f_2, f_4 – injective

f_1 – surjective.

ii) f_3 is surjective.

We want to prove that given $b_3 \in B_3$ $\exists$ some element $a' \in A_3$ s.t. $f_3(a') = b_3$ and want to find conditions on f_i's $i \neq 3$ which make this happen.

Even if we want to use commutativity or exactness, we have to reach the top row from the bottom row.

Now $j_3(b_3) \in B_4$ so to get to the top row, we assume that f_4 is surjective. Thus $\exists$ $a_4 \in A_4$ s.t.

$$f_4(a_4) = j_3(b_3)$$

Now by exactness, $j_3(b_3) \in \operatorname{Im} j_3 \Rightarrow j_3(b_3) \in \ker j_4$

$\Rightarrow$ $j_4 j_3(b_3) = 0 = j_4 f_4(a_4) = f_5 i_4(a_4)$ (by commutativity)

Thus $f_5 i_4(a_4) = 0$

so we assume that f_5 is injective to get

$i_4(a_4)=0. \implies a_4 \in \ker i_4 \implies a_4 \in \operatorname{im} i_3$

so $\exists\ a_3 \in A_3$ w/ $i_3(a_3) = a_4$

So we have two elements in B_3, b_3 and $f_3(a_3)$.

we have

$$\begin{aligned} j_3(b_3 - f_3(a_3)) &= j_3(b_3) - j_3 f_3(a_3) \\ &= f_4(a_4) - f_4 i_3(a_3) \\ &= f_4(a_4) - f_4(a_4) = 0. \end{aligned}$$

Thus $b_3 - f_3(a_3) \in \ker j_3 \implies$ by exactness $\exists\ b_2 \in B_2$

w/ $j_2(b_2) = b_3 - f_3(a_3)$. or $b_3 = f_3(a_3) + j_2(b_2)$

So we can already see a glimpse of how to get a' s.t.

$b_3 = f_3(a')$. We have one element $f_3(a_3)$, so

somehow we need to write $j_2(b_2)$ in $\operatorname{im} f_3$.

So we assume that f_2 is surjective. Then

$\exists\ a_2 \in A_2$ s.t. $f_2(a_2) = b_2$

Thus
$$\begin{aligned} f_3(a_3 + i_2(a_2)) &= f_3(a_3) + f_3 i_2(a_2) \\ &= f_3(a_3) + j_2 f_2(a_2) \\ &= f_3(a_3) + j_2(b_2) \\ &= b_3 \end{aligned}$$

Thus $a_3 + i_2(a_2) = a'$ and f_3 is surjective. So the requirements are

f_2, f_4 surjective and f_5 injective.

iii) bijective.

If you have understood the previous parts then you should really attempt this part on your own.

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