WS 2021/2022

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Topology II

Exercise sheet 10

Exercise 1.
Compute the cohomology groups with $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$ and $\mathbb{R}$ coefficients of $\mathbb{R}P^n$, $\mathbb{R}P^\infty$ and closed surfaces using the universal coefficient theorem.

Exercise 2.
Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle, $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with $\mathbb{Z}_2$-coefficients, with $\mathbb{Z}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 3.

(a) Compute the cohomology groups with arbitrary coeffcients of S^n in two ways:

- via the long exact sequence of a pair in cohomology, and
- via the Mayer–Vietoris sequence for cohomology.

(b) Compute the cohomology groups of all closed surfaces via the Mayer–Vietoris sequence for cohomology.

Exercise 4.

(a) Prove Lemma 6.5 from the lecture.

(b) Is Ext symmetric?

(c) Compute $\operatorname{Ext}(A, B)$ for finitely generated abelian groups A and B.

p.t.o.

Bonus exercise 1.

(a) $\operatorname{Hom}(\mathbb{Z}, G)$ is isomorphic to G for any abelian group G.

(b) $\operatorname{Hom}(\mathbb{Z}_n, \mathbb{Z}_m)$ is isomorphic to $\mathbb{Z}_{\gcd(n,m)}$.

(c) Compute $\operatorname{Hom}(A, B)$ for finitely generated abelian groups A and B.

(d) Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups and let G be another abelian group. If C is free abelian, then the dual sequence

$$0 \longrightarrow \operatorname{Hom}(C, G) \longrightarrow \operatorname{Hom}(B, G) \longrightarrow \operatorname{Hom}(A, G) \longrightarrow 0$$

is also exact.

(e) Does the dual sequence also split?

Bonus exercise 2.

(a) Let $A \subset X$ be a closed subspace that is a deformation retract of some open neighborhood U. Then $H^k(X, A; G)$ is isomorphic to $\widetilde{H}^k(X/A; G)$ induced by the projection map $X \to X/A$.

(b) If A is a retract of X, then $H^k(X; G)$ is isomorphic to $H^k(A; G) \oplus H^k(X, A; G)$.

Bonus exercise 3.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.

(b) Conclude that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.

(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.
Hint: For this you will need the axiom of choice.

(d) Conclude that there exist topological spaces X and Y such that $H^k(X)$ is isomorphic to $H^k(Y)$ for all k, but such that $H_1(X)$ and $H_1(Y)$ are **not** isomorphic. In particular, the roles of homology and cohomology in Corollary 6.6 are not symmetric.
Hint: In [J. Wiegold, *Ext(Q, Z) is the additive group of real numbers*, Bull. Aust. Math. Soc. **1** (1969), 341–343] it is proven that $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ is isomorphic to $\mathbb{R}$. Use this (without proof) together with the universal coefficient theorem for cohomology and Exercise 4 from Sheet 7 to do the exercise.

This sheet will be discussed on Wednesday 12.1. and should be solved by then.