WS 2021/2022 Marc Kegel
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Topology II

Exercise sheet 12

Exercise 1.
We consider the ideal I in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ generated by x_i^2 and $x_i x_j + x_j x_i$ for all $i, j = 1, \dots, n$. The **exterior algebra** $\Lambda[x_1, \dots, x_n]$ is defined to be $\mathbb{Z}[x_1, \dots, x_n]/I$, where the product is usually denote by $\wedge$ and $\deg(x_j) := 1$.

(a) $\Lambda[x_1, \dots, x_n]$ is a free abelian group of rank 2^n where a basis is given by

$$\left\{ x_{k_1} \wedge \cdots \wedge x_{k_l} \middle| k_1 < \cdots < k_l \right\}.$$

(b) $\Lambda[x_1, \dots, x_n]$ is isomorphic to $\Lambda[x_1, \dots, x_{n-1}] \otimes \mathbb{Z}[x_n]/(x_n^2)$, where $\deg(x_n) := 1$.

(c) The cohomology ring of the n-torus $H^*(T^n)$ is isomorphic to $\Lambda[x_1, \dots, x_n]$.

(d) Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$

by computing the Euler characteristic $\chi(T^n)$ via the alternating ranks of its cohomology groups (using the universal coefficient theorem) and directly via a cell structure of T^n.

Exercise 2.
Compute the cup product on the surface of infinite genus (Sheet 8 Exercise 2).

Exercise 3.
Show that the short exact sequence from the universal coefficient theorem does not split natural.

Exercise 4.
A compact connected manifold M does not retract onto its connected boundary ∂M.
Hint: Consider the long exact sequence of the pair $(M, \partial M)$.

This sheet will be discussed on Wednesday 26.1. and should be solved by then.