WS 2021/2022

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Topology II

Exercise sheet 13

Exercise 1.

(a) Let M be a closed manifold of odd Euler characteristic. Show that there exists no compact manifold W with boundary ∂W homeomorphic to M.

(b) Find a closed manifold M in every even dimension that does not occur as the boundary of a compact manifold.

(c) For every closed manifold M construct a **non-compact** manifold W with boundary ∂W homeomorphic to M.

(d) Can the Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$ appear as the boundary of a compact 3-manifold M?

(e) We call two closed n-manifolds M and N **cobordant** if there exists a compact $(n+1)$-manifold W whose boundary ∂W is the disjoint union of M and N. Show that this defines an equivalence relation.

(f) The connected sum $M \# N$ is cobordant to the disjoint union of M and N and the disjoint union of M and M is **nullcobordant**, i.e. cobordant to the empty set.

(g) Which closed surfaces are cobordant?

(h) The set $\mathfrak{N}_n$ of cobordism classes of closed n-manifolds is an abelian group with the disjoint union as operation and the cartesian product of manifolds defines a multiplication

$$\begin{aligned} \mathfrak{N}_n \times \mathfrak{N}_m &\longrightarrow \mathfrak{N}_{n+m} \\ \big([N^n], [M^m]\big) &\longmapsto [N^n \times M^m] \end{aligned}$$

which defines a ring structure on

$$\mathfrak{N}_* = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{N}_n.$$

Exercise 2.

Let M be a connected compact n-manifold. Then

$$\dim_{\mathbb{Z}_2} H_k(M; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H_{n-k}(M, \partial M; \mathbb{Z}_2).$$

Bonus exercise.
A map $f\colon X \to Y$ is called **proper** if the preimage of every compact subset of Y is a compact subset of X.

(a) Describe a proper and a non-proper map $f\colon \mathbb{R} \to \mathbb{R}$.

(b) Every map from a non-compact space to a compact space is **not** proper.

(c) Every homeomorphism and every map from a compact space to a Hausdorff space is proper.

(d) A proper map $f\colon X \to Y$ induces a cochain map

$$f^*\colon C_c^k(Y;G) \to C_c^k(X;G)$$

and thus also a well-defined homomorphism

$$f^*\colon H_c^k(Y;G) \to H_c^k(X;G).$$

(e) Homeomorphic topological spaces have isomorphic cohomology groups with compact support.

(f) Show again that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.

Bonus exercise.
Describe an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$ and describe how to get an embedding into $\mathbb{R}^4$ from it.

Hint: The immersion can be described for example by drawing a detailed figure of its image after watching the Youtube video [J. Leys: *The Boy surface*] or reading [R. Kirby: *What is ... Boy's Surface*, Notices of the AMS, **54** (2007), 1306–1307].
Another method is to build an explicit 3-dimensional model, see for example [A. Chéritat: *A model of Boy's surface in constructive solid geometry*, available on his webpage].
If you are not interested in understanding the construction you can alternatively prove that

$$\begin{aligned} f\colon D^2 &\longrightarrow \mathbb{R}^3 \\ w &\longmapsto \frac{1}{g_1^2 + g_2^2 + g_3^2}(g_1, g_2, g_3) \end{aligned}$$

induces an immersion $\mathbb{R}P^2 \to \mathbb{R}^3$, where

$$\begin{aligned} g_1 &:= -\frac{3}{2}\operatorname{Im}\left[\frac{w\left(1-w^4\right)}{w^6+\sqrt{5}w^3-1}\right] \\ g_2 &:= -\frac{3}{2}\operatorname{Re}\left[\frac{w\left(1+w^4\right)}{w^6+\sqrt{5}w^3-1}\right] \\ g_3 &:= \operatorname{Im}\left[\frac{1+w^6}{w^6+\sqrt{5}w^3-1}\right]-\frac{1}{2}. \end{aligned}$$

This sheet will be discussed on Wednesday 2.2. and should be solved by then.