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Topology II

Exercise sheet 2

Exercise 1.

(a) Describe a space-filling curve, i.e. a continuous surjective map

$$[0,1] \to [0,1] \times [0,1].$$

(b) Show that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.

Exercise 2.
An **exact sequence** is a sequence of groups and homomorphisms

$$\cdots \longrightarrow G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \longrightarrow \cdots$$

such that $\operatorname{Im}(\varphi_i) = \ker(\varphi_{i-1})$ holds for all i.

(a) We consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$$

of abelian groups. Then α is an isomorphism and G is isomorphic to Z_6.

(b) A **short exact sequence** of *abelian* groups is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

(i) The following are equivalent:

- There exists an homomorphism $\lambda\colon C \to B$, such that $\beta \circ \lambda = \mathrm{id}_C$.
- There exists an homomorphism $\mu\colon B \to A$, such that $\mu \circ \alpha = \mathrm{id}_A$.

If one of the above conditions is fulfilled we say that the short exact sequence **splits**. Show that then $B \cong A \oplus C$ holds.

(ii) If C is a free abelian group, then any exact sequence of the above form splits.

(c) Any exact sequence of *vector spaces* splits.

Exercise 3.

(a) Every exact sequence of the form

$$0 \to Z_m \to G \to Z_n \to 0,$$

for m and n coprime is split.

(b) For any natural number $n \geq 2$ there exists an exact sequence of the form

$$0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0.$$

Is this sequence split?

(c) Find for $n \geq 2$ two non-isomorphic groups G such that there exists an exact sequence of the form

$$0 \to \mathbb{Z} \to G \to Z_n \to 0.$$

Exercise 4.
Let X be a path-connected space and denote by CX its cone. Show that

$$\pi_k(CX, X) \cong \pi_{k-1}(X)$$

and construct for a given finitely presented group G a pair of path-connected spaces (Y, A) with $\pi_2(Y, A) \cong G$.

Bonus exercise.
We consider the following commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & A_3 & \xrightarrow{i_3} & A_4 & \xrightarrow{i_4} & A_5 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
B_1 & \xrightarrow{j_1} & B_2 & \xrightarrow{j_2} & B_3 & \xrightarrow{j_3} & B_4 & \xrightarrow{j_4} & B_5
\end{array}$$

Find minimal conditions on f_1, f_2, f_4, f_5 (w.r.t. injectivity and surjectivity), that imply that f_3 is

(i) injektive,

(ii) surjektive,

(iii) bijektive.

Show by writing down examples that these conditions cannot be wakened further.

This sheet will be discussed on Friday 3.11. and should be solved by then.