Topology II

Exercise sheet 2

Exercise 1.

(a) Describe a space-filling curve, i.e. a continuous surjective map

$$[0,1] \to [0,1] \times [0,1].$$

(b) Show that \mathbb{R}^n is homeomorphic to \mathbb{R}^m if and only if n = m.

Exercise 2.

An exact sequence is a sequence of groups and homomorphisms

$$\cdots \longrightarrow G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \longrightarrow \cdots$$

such that $\operatorname{Im}(\varphi_i) = \ker(\varphi_{i-1})$ holds for all i.

(a) We consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z} \stackrel{\alpha}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

of abelian groups. Then α is an isomorphism and G is isomorphic to Z_6 .

(b) A **short exact sequence** of abelian groups is an exact sequence of the form

$$0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0.$$

- (i) The following are equivalent:
 - There exists an homomorphism $\lambda \colon C \to B$, such that $\beta \circ \lambda = \mathrm{id}_C$.
 - There exists an homomorphism $\mu \colon B \to A$, such that $\mu \circ \alpha = \mathrm{id}_A$.

If one of the above conditions is fulfilled we say that the short exact sequence **splits**. Show that then $B \cong A \oplus C$ holds.

- (ii) If C is a free abelian group, then any exact sequence of the above form splits.
- (c) Any exact sequence of vector spaces splits.

Exercise 3.

(a) Every exact sequence of the form

$$0 \to Z_m \to G \to Z_n \to 0$$
,

for m and n coprime is split.

(b) For any natural number $n \geq 2$ there exists an exact sequence of the form

$$0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0.$$

Is this sequence split?

(c) Find for $n \geq 2$ two non-isomorphic groups G such that there exists an exact sequence of the form

$$0 \to \mathbb{Z} \to G \to Z_n \to 0.$$

Exercise 4.

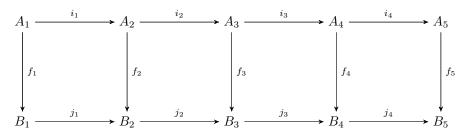
Let X be a path-connected space and denote by CX its cone. Show that

$$\pi_k(CX,X) \cong \pi_{k-1}(X)$$

and construct for a given finitely presented group G a pair of path-connected spaces (Y, A) with $\pi_2(Y, A) \cong G$.

Bonus exercise.

We consider the following commutative diagram of abelian groups with exact rows:



Find minimal conditions on f_1 , f_2 , f_4 , f_5 (w.r.t. injectivity and surjectivity), that imply that f_3 is

- (i) injektive,
- (ii) surjektive,
- (iii) bijektive.

Show by writing down examples that these conditions cannot be wakened further.

This sheet will be discussed on Friday 3.11. and should be solved by then.