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Topology II

Exercise sheet 3

Exercise 1.

We call a *smooth* manifold M **orientable** if M admits an atlas in which all charts are compatible and the determinant of the Jacobi matrices of all transition maps (i.e. $\psi \circ \phi^{-1}$ for charts ψ and ϕ) are everywhere positive.

(a) Show that S^n is orientable by explicitly describing such an atlas.

(b) The Möbius strip is not orientable.

(c) If we glue a 2-disk and a Möbius strip along its boundaries we get a space that is homeomorphic to $\mathbb{R}P^2$.

(d) A surface is orientable if and only if it contains no Möbius strip, i.e. if and only if there exists no closed path interchanging right and left.

Exercise 2.

(a) Show that $\mathbb{R}P^n$ is a closed manifold of dimension n and that $\mathbb{R}P^1$ is diffeomorphic to S^1. For which n is $\mathbb{R}P^n$ orientable?

(b) Show that $\mathbb{C}P^n$ is a closed oriented manifold of dimension $2n$ and that $\mathbb{C}P^1$ is diffeomorphic to S^2.

(c) We denote by $\mathbb{H}$ the quaternions and define the **quaternionic projective spaces** $\mathbb{H}P^n$ as

$$\mathbb{H}P^n := \big(\mathbb{H}^{n+1} \setminus \{0\}\big)/_\sim,$$

where $u \sim v$ if and only if there exists an $h \in \mathbb{H} \setminus \{0\}$ such that $v = hw$. Verify that $\mathbb{H}P^n$ is a well-defined closed oriented manifold of dimension $4n$ and show that $\mathbb{H}P^1$ is homeomorphic to S^4.

Exercise 3.

(a) Prove the 2-dimensional Poincaré conjecture.

(b) More generally, two closed surfaces are homeomorphic if and only if they are homotopy equivalent.

(c) Is the statement in (b) also true for compact surfaces with boundary?

p.t.o.

Exercise 4.

(a) The Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$ carries the structure of an S^1-bundle over S^1.

(c) We identify S^{2n+1} with the unit sphere in $\mathbb{C}^{n+1}$. The map

$$p\colon S^{2n+1} \longrightarrow \mathbb{C}P^n$$
$$(z_0, \cdots, z_n) \longrightarrow [z_0 : \cdots : z_n]$$

is an S^1-bundle.

(d) We identify S^{4n+3} with the unit sphere in $\mathbb{H}^{n+1}$. The map

$$p\colon S^{4n+3} \longrightarrow \mathbb{H}P^n$$
$$(h_0, \cdots, h_n) \longrightarrow [h_0 : \cdots : h_n]$$

is an S^3-bundle.

(e) What conclusion do we get from the above bundles about the homotopy groups of these spaces?

Exercise 5.
Let $X_1 \subset X_2 \subset X_3 \subset \cdots$ be an infinite sequence of inclusions of topological spaces. We define the limit

$$X_\infty := \varinjlim X_i := \bigcup_{i \in \mathbb{N}} X_i,$$

where a set U in X_∞ is called open if $U \cap X_i$ is open in X_i for all $i \in \mathbb{N}$.

If we apply the above construction to the sequence $S^0 \subset S^1 \subset S^2 \subset \cdots$ we get the space S^∞ and from the sequence $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots$ we get the spaces and $\mathbb{C}P^\infty$.

(a) $\pi_k(S^\infty) = 0$ for all $k \geq 1$.

(b) Define an S^1-bundle $p\colon S^\infty \to \mathbb{C}P^\infty$ in analogy to Exercise 2(b).

(c) Compute from the associated long exact sequence the homotopy groups of $\mathbb{C}P^\infty$ and conclude that S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups.

Bonus exercise.

(a) Classify compact 1-manifolds (possibly with boundary).

(b) Workout the details from the proof sketch of the classification theorem of surfaces.
Hint: It might be helpful to have a look at Chapter 5 of:
`https://www.mathematik.hu-berlin.de/~kegemarc/19SSTopologie/Skript.pdf`

This sheet will be discussed on Wednesday 10.11. and should be solved by then.