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Topology II

Exercise sheet 4

Exercise 1.
Compute the simplicial homology groups of the n-sphere, the Möbius strip and the Klein bottle using directly the definition of simplicial homology.

Exercise 2.
Compute the degree of the constant map $\mathrm{c}\colon S^n \to S^n$, the identity $\mathrm{Id}\colon S^n \to S^n$ and the antipodal map $\text{-Id}\colon S^n \to S^n$.

Exercise 3.
An oriented q-simplex $\sigma = (x_0, \ldots, x_q)$ **induces** an orientation on any of its $(q-1)$-dimensional faces τ via

$$\tau = (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_q).$$

We call a triangulated n-manifold **orientable**, if there exists orientations on the n-simplices such that any two adjacent n-simplices induce opposite orientations on their common $(n-1)$-dimensional face.

(a) Draw sketches in Dimensions 2 and 3.

(b) Show that this definition of orientability coincides with the definition for smooth manifolds from the last sheet.

(c) Let M be a smooth closed n-manifold. Show that

$$H_n(M) \cong \begin{cases} \mathbb{Z}\,; & \text{if } M \text{ is orientable,} \\ 0\,; & \text{if } M \text{ is not orientable.} \end{cases}$$

Hint: It might be helpful to work with simplicial homology and start with an explicit triangulation of the 2-torus and to identify an explicit 2-cycle generating the second homology. Next, one can consider the Klein bottle. Does there exists a 2-cycle on the Klein bottle? Finally, try to generalize these arguments.

Exercise 4.
Let X be the topological space consisting of a single point. Compute all singular homology groups of X directly from the definitions. For which other spaces can we compute all singular homology groups?

Exercise 5.

(a) Find a way to relate the (singular) homology group of a topological space X to the (singular) homology groups of its path-connected components.

(b) Let X be a path-connected space. Show that $H_0(X) \cong \mathbb{Z}$ and that $H_1(X)$ is isomorphic to the abelization $\pi_1^{ab}(X)$ of the fundamental group of X.

This sheet will be discussed on Wednesday 17.11. and should be solved by then.