WS 2021/2022

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Topology II

Exercise sheet 6

Exercise 1.

(a) Reformulate Corollary 3.10 using reduced homology groups.

(b) Compute again the homology groups of spheres by using Theorem 3.13.

(c) Compute the homology groups of all closed surfaces.
Hint: You can do this by using the Mayer–Vietoris sequence or by using Theorem 3.13.

Exercise 2.
Construct, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, a map $f\colon S^n \to S^n$ of degree k.
Hint: Construct from a given map $S^{n-1} \to S^{n-1}$ a map $S^n \to S^n$ of the same degree by identifying S^n with the suspension ΣS^{n-1}.

Exercise 3.
Let K be a knot in S^3, i.e. the image of an embedding of S^1 into S^3. Compute all homology groups of $S^3 \setminus K$.
Bonus: What can you say about the homotopy groups of $S^3 \setminus K$?

Exercise 4.
Let $x_0 \in X$ be a point.

(a) $\widetilde{H}_k(X) \cong H_k(X)$ for $k \geq 1$.

(b) $\widetilde{H}_0(X) \cong \mathbb{Z}^{n-1}$, where n is the number of path components of X.

(c) $\widetilde{H}_k(X) \cong H_k(X, \{x_0\})$.

Exercise 5.
Prove the invariance of the domain:
Let $U, V \subset S^n$ be homeomorphic. Then U is open if and only if V is open.
Hint: Use Proposition 3.21 and Theorem 3.17.

Bonus exercise.

(a) Prove a version of the Mayer–Vietoris sequence for simplicial homology groups.

(b) Present a geometric description of the connecting homomorphism in the Mayer–Vietoris sequence.

Bonus exercise.
Construct a pair of spaces (X, A) such that $\widetilde{H}_k(X/A)$ is **not** isomorphic to $H_k(X, A)$.

Bonus exercise.

(a) Describe explicitly a sequence of embeddings $f_k \colon D^3 \to S^3$ converging to an embedding $f \colon D^3 \to S^3$, such that $f(S^2)$ is homeomorphic to Alexander's horned sphere.

(b) Show that the complement of $f(D^3)$ is not contractible by showing that it admits a non-vanishing element γ in $\pi_1(S^3 \setminus f(D^3))$.

(c) Show explicitly that γ is nullhomologous. (Explicitly means here without using Propositi-on 3.21.)

(d) Can we construct an embedding of S^2 into S^3 such that both components of the complement are not contractible?

Hint: For some help in this exercise one could have a look at page 170 in Hatcher's book on algebraic topology.

This sheet will be discussed on Wednesday 1.12. and should be solved by then.