WS 2021/2022

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Topology II

Exercise sheet 7

Exercise 1.

(a) Describe CW-decompositions of all closed surfaces.

(b) The n-torus $T^n := S^1 \times \cdots S^1$ admits the structure of a CW-complex.

(c) Describe CW-complexes on $\Sigma_g \times \Sigma_h$.
Hint: Use (a) and the technique from (b).

Exercise 2.

We define the Euler characteristic $\chi(X)$ of a finite CW-complex X of dimension n to be

$$\chi(X) := \sum_{k=0}^{n} (-1)^k |I_k|,$$

where $|I_k|$ denotes the number of k-cells in X.

(a) Compute the Euler characteristic for your favorite CW-structure of S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and Σ_g.

(b) Show that the Euler characteristic of a CW-complex only depends on the homotopy type of X and not on the particular CW-structure.
Hint: Relate the Euler characteristic of a CW-complex to its cellular homology groups.

(c) Let K and L be CW-complexes that intersect in a common subcomplex $K \cap L$. Verify the gluing formula for the Euler characteristic:

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L).$$

(d) Let X and Y be finite CW-complexes. Find a way to compute $\chi(X \times Y)$ from the Euler characteristics of X and Y.

(e) The Euler characteristic together with the orientability is a complete invariant of closed surfaces.

(f) The 2-sphere admits a CW-decomposition with an arbitrary number of even cells, but no CW-decomposition with an odd number of cells.

p.t.o.

Exercise 3.
Let M be a smooth compact n-manifold. We define $\Delta(M)$ as the minimal number of n-simplices in a triangulation of M. We define similarly $c(M)$ as the minimal number of cells in a cell decomposition of M.

(a) Compute Δ and c for S^2, $\mathbb{R}P^2$ and T^2.

(b) What can you say about Δ and c for other surfaces?

Exercise 4.

(a) $\mathbb{R}^\infty$ is homeomorphic to $\mathbb{C}^\infty$.

(b) Choose an explicit CW-structure on S^∞, describe the corresponding cellular chain complex and compute the cellular homology of S^∞.

(c) S^∞ is contractible.

Bonus exercise.
Let X be a finite connected CW-complex. How can the fundamental group of X be computed?
Bonus: What can be said if X is **not** finite?

This sheet will be discussed on Wednesday 8.12. and should be solved by then.