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Topology II

Exercise sheet 9

Exercise 1.
In the lecture we have seen four different ways to compute homology groups with coefficients:

- via the Mayer–Vietoris sequence,
- directly from the definition and a CW-structure,
- with the Bockstein homomorphism or
- using the universal coefficient theorem.

Compute the homology groups of $\mathbb{R}P^n$ and the Klein bottle with $\mathbb{Q}$- and $\mathbb{Z}_2$-coefficients using as many of the above methods as possible.
Bonus: Do the same for $\mathbb{Z}_p$- and $\mathbb{R}$-coefficients.

Exercise 2.

(a) Compute the homology of the n-torus $T^n := S^1 \times \cdots \times S^1$.

(b) Let M and N be closed **topological** manifolds. Show that $M \times N$ is orientable if and only if M and N are orientable.

Exercise 3.

(a) A short exact sequence

$$0 \to B \to C \to D \to 0$$

of abelian groups induces an exact sequence of the form

$$0 \to \operatorname{Tor}(A, B) \to \operatorname{Tor}(A, C) \to \operatorname{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

(b) Prove Lemma 5.5 from the lecture.

Exercise 4.
Show that the short exact sequence from the universal coefficient theorem does **not** split natural.
Hint: Consider the projection map

$$f \colon \mathbb{R}P^2 \cong D^2 \cup \mathbb{R}P^1 \longrightarrow \mathbb{R}P^2/\mathbb{R}P^1 \cong S^2.$$

Exercise 5.
Let $\mathbb{F}$ be an arbitrary field and X a finite CW-complex of dimension n. Then the Euler characteristic of X is given by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{F}} H_k(X, \mathbb{F}).$$

Bonus exercise 1.
A map $f\colon X \to Y$ induces an isomorphism on homology with $\mathbb{Z}$-coefficients if and only if f induces an isomorphism on homology with $\mathbb{Q}$-coeffcients and $\mathbb{Z}_p$-coeffcients for all primes p.

Bonus exercise 2.

(a) Let A be a finitely generated abelian group, i.e. A is of the form $\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$ for natural numbers a_i. For B isomorphic to $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ we have

$$A \otimes B \cong B^r.$$

(b) For finitely generated abelian groups A and B we have

$$\mathrm{rk}(A \otimes B) = \mathrm{rk}(A) \cdot \mathrm{rk}(B).$$

(c) Let R be a commutative ring. Then

$$A \longmapsto A \otimes R$$
$$(f\colon A \to B) \longmapsto (f \otimes \mathrm{id}\colon A \otimes R \to B \otimes R)$$

defines a functor from the category of abelian groups to the category of R-modules.
Hint: If you are unfamiliar with the notions of rings and modules, it may be helpful to first consider the case that R is a field. (Then R-modules are just the R-vector spaces.)

Bonus exercise 3.
Classify finitely generated abelian groups, i.e. prove that any finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}$$

for natural numbers a_i.

Bonus exercise 4.
$\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup of A.

This sheet will be discussed on Wednesday 5.1. and should be solved by then.