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Differentialgeometrie I

Exercise sheet 11

Exercise 1.

(a) $\mathbb{R}^n$, S^n, and $\mathbb{H}^n$ are complete Riemannian manifolds.

(b) Give an example of a non-complete connected Riemannian manifold M such that any two points p and q can be joined by a distance realizing geodesic in M.

(c) Let M be a complete Riemannian manifold and let $N \subset M$ be a closed embedded submanifold. Then N with the induced metric is again complete.

(d) Let M be a compact Riemannian manifold. Show that M has finite diameter, and that any two points $p, q \in M$ can be joined by a geodesic of length $d(p, q)$.

Exercise 2.

(a) Show that a Riemannian manifold is complete if and only if it satisfies the Heine–Borel property, i.e. a set is compact if and only if it is bounded and closed.

(b) Let M be a Riemannian manifold. A curve $\gamma\colon [0, a) \to M$ is called *divergent*, if for every compact set $K \subset M$ there exists a $t_0 \in [0, a)$ such that $\gamma(t) \notin K$ for all $t > t_0$. Show: M is complete if and only if all divergent curves are of infinite length.

Exercise 3.

Determine a formula for the second variation of the length, i.e prove Lemma 6.40 from lecture. Determine also a formula for the second variation of the energy.