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Differentialgeometrie I

Exercise sheet 12

Exercise 1.

(a) Let $X = U \cup V$ for open and simply connected sets U and V such that $U \cap V$ is path-connected. Then X is also simply connected.
Hint: Use the Lebesgue covering theorem (Lemma 1.3) to decompose the image of a loop into paths that are entirely in U or entirely in V.

(b) Conclude that S^n is simply connected for $n \geq 2$.

(c) $\mathbb{R}^n \setminus \{0\}$ is simply connected for $n \geq 3$.

(d) Let X and Y be path-connected. Then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

(e) Compute the fundamental group of $S^n \times S^m$ for $n, m \geq 2$.

Exercise 2.

(a) Let $f\colon (X, x_0) \to (X, x_0)$ be a continuous map homotopic to the identity (not necessary rel $\{x_o\}$). Show that $f_\star\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ is an inner automorphism, i.e. $f_\star$ is of the form

$$f_\star[u] = [v]^{-1}[u][v]$$

for a suitable loop v at the point x_0.

(b) Let F be a homotopy between $f, g\colon X \to Y$. Let x_0 be a base point in X and u be the path $u(t) = F(x_0, t)$ from $f(x_0)$ to $g(x_0)$. Then

$$f_\star = u_\#^{-1} g_\star\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

Exercise 3.
Fill in the details of the proofs of Theorem 7.4 and Theorem 7.6 from the lecture.

p.t.o.

Bonus exercise 1.

(a) A path-connected space is connected.

(b) $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.
Remark: We have seen that $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^m$ if and only if $n = m$. For that we can just study the differential of a smooth map, which is a vectorspace homomorphism of the tangent spaces. On the other hand, the same statement up to homeomorphism is not trivial at all and needs deep methods from algebraic topology. Once we have computed the fundamental group of S^1 to be non-trivial, we will be able to show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

(c) Descibe a space filling curve, i.e. a continous surjective map

$$[0,1] \longrightarrow [0,1] \times [0,1].$$

(d) Describe a space which is connected but not path-connected.

(e) A connected manifold is also path-connected.