WS 2022/23

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Differentialgeometrie I

Exercise sheet 13

Exercise 1.

(a) Describe an operation of $\mathbb{Z}$ on $\mathbb{R} \times [0,1]$ which has the Möbius strip as orbit space. Use this operation to compute the fundamental group of the Möbius strip.

(b) Describe three different operations of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on the 2-torus with orbit spaces cylinder, 2-sphere, and Klein bottle, respectively.

Exercise 2.
Let u and v be loops in a topological group $(G, \circ)$ with base point e. Let $u \circ v$ be the loop defined by $(u \circ v)(s) := u(s) \circ v(s)$. Show that

$$uv \simeq u \circ v \simeq vu \text{ rel}\{0,1\}$$

and conclude that $\pi_1(G, e)$ is abelian.

Exercise 3.
Determine the universal cover of the Klein bottle and describe the fundamental group of the Klein bottle. Can the Klein bottle carry the structure of a topological group?

Exercise 4.
The goal of this exercise is to prove Theorem 7.11. Let X be simply connected and G be a topological group acting discretely on X. Show:

(a) $\pi \colon X \to X/G$ is a covering.

(b) The fundamental group $\pi_1(X/G)$ is isomorphic to G.
Hint. Proceed analogously to the proof of Theorem 7.10.

p.t.o.

Exercise 5.
Let Σ_g be a surface of genus $g \geq 2$ and $\mathbb{H}^2$ the hyperbolic plane.

(a) Describe a group action by isometries on $\mathbb{H}^2$ whose quotient is Σ_g.
Hint: Try to generalize the construction of the universal cover of T^2. First show that Σ_g can be obtained from a regular $2g$-gon by identifying edges. Then show that you can cover the hyperbolic plane by isometric $2g$-gons that are disjoint or only intersect in their pieceweise geodesic boundaries. Finally, consider the group of these isometries.

(b) Use this group action to deduce that Σ_g carries a metric of constant negative curvature.
Hint: Use Exercise 1 (e) from Sheet 9.

(c) Compute the fundamental group of Σ_g.
Hint: Show that the group acts discretely and use Exercise 4.

(d) Show that Σ_g is not homeomorphic to Σ_h, for $g \neq h$.
Hint: Compute the abelization of the fundamental group. Compare your result to the Euler characteristic of the surface.

Bonus exercise 1.
Prove the lifting property for homotopies of paths, i.e. prove Lemma 6.9 from the lecture.
Hint: Proceed analogously to the proof of Lemma 6.8.