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Differentialgeometrie I

Exercise sheet 3

Exercise 1.

(a) Compute the first and second fundamental forms of a graph and determine its Christoffel symbols both extrinsically (i.e. by using the definition) as well as intrinsically (i.e. with Theorem 3.5).

(b) Compute the matrix (L_{ij}) describing the second fundamental form of a rotation surface and show that $\det(L_{ij})$ is vanishing if and only if every meridian is a straight line.
Hint: In Exercise 2 of Sheet 2 we have computed its metric coefficients. Try to visualize the second statement in a picture.

Exercise 2.
Consider the 2-sphere S^2 (without the zero meridian) parametrized as in the lecture by

$$x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

(a) Calculate the metric tensor, its inverse matrix, the coefficients L_{ij} of the second fundamental form and the Christoffel symbols Γ_{ij}^k. Justify geometrically why none of these quantities depend on φ.

Let γ be a curve parametrized by arc length with trace on S^2.

(b) The normal curvature k_n of γ is constant. What is its value? Does it depend on the radius of the sphere?

(c) If the geodesic curvature k_g of γ is constant then γ is a circle.

(d) If γ is a geodesic then γ is a great circle.

(e) Determine the geodesic curvature of a latitudinal circle.
Hint: In case you are missing the proper education in geography, a latitudinal circle is in the above parametrization given by $\varphi \mapsto x(\theta_0, \varphi)$ for a fixed value θ_0. For all these exercises I suggest (in addition to your other arguments) to create instructive sketches visualizing the situation.

Exercise 3.

(a) Let M be a surface and E a plane in $\mathbb{R}^3$ that intersects M in a curve γ. Then γ is a geodesic if E is a symmetry plane of M.

(b) Every straight line in $\mathbb{R}^3$ contained in a surface M is a geodesic.

(c) LetM_1 be the surface $\{x^2 + y^2 - z^2 = 1\}$ and M_2 the surface $\{z = x^2 - y^2\}$. Draw detailed pictures of M_1 and M_2 and describe geodesics on M_1 and M_2.

Exercise 4.

Consider the upper half plane

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$$

with the so-called **hyperbolic metric** given by $g_{11} = g_{22} = 1/y^2$, $g_{12} = g_{21} = 0$. We will show in the second part of the lecture that it is impossible to realize $\mathbb{R}^2_+$ with that metric as a surface in $\mathbb{R}^3$. Nevertheless all intrinsic calculations can be carried out with respect to this metric.

(a) Compute the Christoffel symbols.

(b) Determine the geodesics γ and α with

$$\begin{aligned}
\big(\gamma^1(0), \gamma^2(0)\big) &= (x_0, 1), \\
\big((\gamma^1)'(0), (\gamma^2)'(0)\big) &= (0, 1), \\
\big(\alpha^1(0), \alpha^2(0)\big) &= (a, r), \text{ and} \\
\big((\alpha^1)'(0), (\alpha^2)'(0)\big) &= (r, 0).
\end{aligned}$$

Find explicit parametrizations of γ and α and describe both geometrically in $\mathbb{R}^2_+$.

(c) Let $X_0 = (0, 1)$ be a tangent vector at the point $(0, 1)$ of $\mathbb{R}^2_+$. Verify that X_0 is a unit vector in $T_{(0,1)}\mathbb{R}^2_+$ with respect to the hyperbolic metric. Let $X(t)$ be the parallel transport of X_0 along the curve $x = t$, $y = 1$. Show that the angle between $X(t)$ and the y-axis is equal to t.

Exercise 5.

Determine the transformation behavior of the second fundamental form L_{ij} and the Christoffel symbols Γ^k_{ij} under a coordinate transformation.