WS 2022/23 Marc Kegel
Naageswaran Manikandan

Differentialgeometrie I

Exercise sheet 4

Exercise 1.
The 2-torus T^2 is the surface in $\mathbb{R}^3$, which is obtained by rotating the circle $(r-2)^2+z^2=1$ in the (r,z)-plane around the z-axis.

(a) Sketch this in a figure and use that figure to visualize the following exercises.

(b) The 2-torus (except for one meridian and one longitude) can be parametrized by

$$x(u,v)=\big((2+\cos u)\cos v,(2+\cos u)\sin v,\sin u\big),(u,v)\in(0,2\pi).$$

(c) Determine the Gaussian curvature and the mean curvature.

Exercise 2.
Calculate the first fundamental form, the second fundamental form, and the various curvatures of the helicoid

$$x(u,v)=(v\cos u,v\sin u,cu)$$

and of the catenoid

$$x(u,v)=(a\cosh v\cos u,a\cosh v\sin u,av).$$

Here a and c are positive real constants. Are these surfaces (locally) isometric? Draw sketches of these surfaces and describe geodesics on them.

Exercise 3.
The converse of the theorema egregium is false, i.e. a diffeomorphism $f\colon M\to N$ between two surfaces, which satisfies $K_M(p)=K_N(f(p))$ for all $p\in M$, needs not be a local isometry.
Hint: Consider the parametrized surfaces

$$x(t,\varphi)=(t\cos\varphi,t\sin\varphi,\log t),\text{ and}$$
$$y(t,\varphi)=(t\cos\varphi,t\sin\varphi,\varphi),\text{ for }(t,\varphi)\in R_+\times(0,2\pi).$$

Exercise 4.
Let $x\colon U\to\mathbb{R}^3$ be a parametric piece of a surface. A piece of a surface **parallel** to x is given by

$$y(u,v)=x(u,v)+c\,n(u,v),$$

where c is a constant and n is the normal vector of $x(U)$. At which points is y regular? Express the Gaussian and mean curvature of y at all its regular points in terms of the curvatures of x.

p.t.o.

Exercise 5.

(a) The isometries $f\colon M \to M$ of a surface M form a group in a natural way. This group is called the isometry group of M.

(b) The isometry group of S^2 is the group of orthogonal (3×3)-matrices.

A diffeomorphism $f\colon M \to N$is called a **conformal** map if

$$\langle df(X), df(Y)\rangle_{f(p)} = \lambda(p)\langle X, Y\rangle_p$$

for all $p \in M$ and $X, Y \in T_pM$, where $\lambda\colon M \to R_+$ is a differentiable function. Analogously to the notion of local isometry, we define local conformal maps.

(c) S^2 is not locally isometric, but locally conformal to the plane.

Exercise 6.

(a) Let γ be a curve parameterized by arc length on a surface M, and let S be the intrinsic normal along γ. Then S is parallel along γ if and only if γ is a geodesic.

(b) Let γ be as in (a), with non-vanishing curvature. Let X_N be the component of N tangent to M. Show that $X_N = N - \langle N, n\rangle n$, and that the following statements are equivalent:

  (i) $X_N \equiv 0$,

  (ii) γ is a geodesic,

  (iii) X_N is parallel along γ.

Exercise 7.
Let α be a curve with trace on a surface M. Write $n(t)$ for the normal vector of M in $\alpha(t)$. Necessary and sufficient for α to be a curvature line of M is

$$\dot{n}(t) = \lambda(t)\dot{\alpha}(t)$$

with a differentiable function $\lambda(t)$, which is except for the sign the corresponding principal curvature in $\alpha(t)$.