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Differentialgeometrie I

Exercise sheet 6

Exercise 1.
An n-gon is a piecewise smooth, regular curve on a surface M, which boundas a disk in M and whose n smooth segments are geodesics in M.

(a) Let M be a surface with $K \leq 0$. Then there is no n-gon for $n = 0, 1, 2$. (A 0-gon is a closed geodesic bounding a disk in M.)

(b) Find an example of a surface with $K < 0$ on which there exists a closed geodesic.

Exercise 2.
Consider the Poincaré disk $U = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ with the metric given by

$$\big(g_{ij}(u, v)\big) = \begin{pmatrix} 4/(1-u^2-v^2)^2 & 0 \\ 0 & 4/(1-u^2-v^2)^2 \end{pmatrix}.$$

As in Exercise 4 of Sheet 3, this is an example of an abstractly defined surface. We call this metric the **hyperbolic metric** on U.

(a) Compute its Christoffel symbols and its Gaussian curvature.

(b)) The diameters of U and circle arcs which orthogonally intersect the unit circle $u^2 + v^2 = 1$ are geodesics. Conversely, every geodesic is of this form.

(c) Determine the area of an n-gon whose vertices lie on the unit circle.

(d) Find an isometry from the Poincaré disk to the upper half plane (from Exercise 4 of Sheet 3). *Hint:* Use complex coordiantes to describe the map.

Exercise 3.

(a) Show that $(0, 0)$ is a zero of the following vector fields, compute its index and visualize that computations in local sketches:

  (i) $X(u, v) = (u, v)$,
  (ii) $X(u, v) = (-u, v)$,
  (iii) $X(u, v) = (u, -v)$,
  (iv) $X(u, v) = (u^2 - v^2, -2uv)$,
  (v) $X(u, v) = (u^3 - 3uv^2, v^3 - 3u^2v)$.

(b) Can it happen that the index of a zero is vanishing? If yes, give an example.

(c) What can be said about the existence of vector fields without zeros on non-compact surfaces?

Exercise 4.

(a) Let $\gamma\colon [a,b] \to M$ be a curve parameterized by arc length in a surface $M \subset \mathbb{R}^3$. Let α be the angle between the normal vector n of M and the binormal vector B of the curve γ. Show that the curvature k of γ (as a space curve) and the geodesic curvature k_g are related by $k_g = k \cos\alpha$.

(b) Let γ be the circle of latitude $\phi_0 \in [-\pi/2, \pi/2]$ on the 2-sphere S^2. Show, that $k = 1/\sin\phi_0$ and deduce that $k_g = \cot\phi_0$.

(c) Use the local theorem of Gauss–Bonnet to show that the subsurface of S^2 whose positive boundary is γ has area $2\pi(1 - \cos\phi_0)$.

Exercise 5.

(a) The 2-sphere is orientable.

(b) The Möbius strip is not orientable.

(c) Let N be an orientable surface and $f\colon M \to N$ be a smooth map that is a local diffeomorphism around every point $p \in M$. Then M is also orientable.

Bonus exercise 1.
The boundary of a Möbius strip is diffeomorphic to S^1. Construct an embedding of the Möbius strip into $\mathbb{R}^3$ such that its boundary gets mapped to a standard round circle. Create a paper model.

Bonus exercise 2.
Let R_l be a paper rectangle with boundary lengths 1 and l. Perform experiments by creating paper models for various values of l to find a conjecturally maximal length l such that this is possible.
Challenge: Prove that your bound on l is indeed optimal.