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Differentialgeometrie I

Exercise sheet 7

Exercise 1.

(a) The atlas of S^n defined in the lecture defines a smooth structure on S^n.

(b) Describe an atlas with exactly two charts by using stereographic projections.
Hint: Try to generalize Exercise 1 from Sheet 2.

(c) Show that the smooth structures from (a) and (b) are the same.

(d) Is there an atlas of S^n with a single chart?

Exercise 2.

(a) The unit cube
$$W^n = \{x \in \mathbb{R}^{n+1} \big| \max\{|x_1|, \dots, |x_{n+1}|\}\}$$
is not a smooth submanifold of $\mathbb{R}^{n+1}$ but carries the structure of a smooth abstract manifold.

(b) Is the union of two lines in $\mathbb{R}^2$ a manifold?

Exercise 3.
The **complex projective space** $\mathbb{C}P^n$ is the quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ under the diagonal action of the group $S^1 \subset \mathbb{C}$, i.e.

$$\mathbb{C}P^n = \{[x_0 : \dots : x_n] \big| (x_0, \dots, x_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}\},$$

where $[x_0 : \dots : x_n] = [y_0 : \dots : y_n]$ holds exactly if there exists a $\lambda \in S^1$ with $(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$.

(a) $\mathbb{C}P^n$ is a smooth $2n$-manifold.

(b) $\mathbb{C}P^1$ is diffeomorphic to S^2.
Hint: In the lecture we have seen that $\mathbb{R}P^1$ is homeomorphic to S^1. Show first that S^1 is diffeomorphic to $\mathbb{R}P^1$ by writing down explicitly a diffeomorphism. Try to generalize this construction.

(c) $\mathbb{R}P^3$ is homeomorphic to $SO(3)$. Are these spaces also diffeomorphic?

p.t.o.

Exercise 4.
If $f\colon M \to N$ and $g\colon N \to Q$ are smooth maps between smooth manifolds, then

$$T_p(g \circ f) = T_{f(p)}g \circ T_p f.$$

Exercise 5.
Let $f\colon M \to N$ a smooth map between smooth manifolds. We say that a point $q \in N$ is a **regular value** of f if for all points p in $f^{-1}(q)$ the Jacobi matrix of f has full rank in local coordinates around p.

(a) Show that this is well-defined, i.e. does not depend on the choice of local coordinates.

(b) If q is a regular value of M then $f^{-1}(q)$ is a smooth submanifold of M. What is its dimension?
Hint: Generalize the regular value theorem for smooth maps $\mathbb{R}^m \to \mathbb{R}^n$.

(c) The special orthogonal group $SO(n)$ is a smooth submanifold of the space of all $(n \times n)$-matrices. What is its dimension?
Hint: The space of all $(n \times n)$-matrices can be identified with $\mathbb{R}^{n^2}$. It might be helpful to first show that the general linear group $GL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are submanifolds.

Bonus exercise 1.

(a) Construct a topological space with countable basis that is locally homeomorphic to $\mathbb{R}$, but **not** Hausdorff.

(b) Construct a topological Hausdorff space that is locally homeomorphic to $\mathbb{R}$, but which has **no** countable basis.

(c) We denote by M the real line with its standard topology. Let $h\colon M \to \mathbb{R}$ be the map given by $h(x) = x^3$. Show that h is a homoeomorphism and explain how h (and any other such homeomorphism) defines a smooth structure on M. Show that that h induces a different smooth structure on M than the identity map, but show that the resulting smooth manifolds are diffeomorphic.

Bonus exercise 2.
Let X be a topological space and $M \subset X$ a subset with its **subspace topology**, i.e. $U \subset M$ is open if there exists an open set $V \subset X$ such that $U = V \cap M$.

(a) Verify that the subspace topology defines a topology.

(b) Verify that the quotient topology (as defined in lecture) is a topology.

(c) The quotient topology is the finest topology (i.e. the topology with the most open sets), for which the quotient map is continuous.